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Asymptotic classes of finite Moufang polygons

Pietro Dello Stritto

University of Mons – UMONS, Institut de Mathematique, Batiment Le Pentagone, Avenue du Champ de Mars, 6, 7000 Mons, Belgium

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ABSTRACT

In this paper we study the model theory of classes of finite Moufang polygons. We show that each family of finite Moufang polygons forms an 'asymptotic class'. As a result, since every non-principal ultraproduct of an asymptotic class is 'measurable', and therefore supersimple of finite rank, we obtain examples of (infinite) supersimple Moufang polygons of finite rank.

In a forthcoming paper, [8], we will show that all supersimple Moufang polygons of finite rank arise over supersimple fields and belong to exactly those families which also have finite members.

This body of work will give a description of groups with supersimple finite rank theory which have a definable spherical Moufang *BN*-pair of rank at least two.

1. Introduction

There is a well-known model-theoretic conjecture, often referred to as the 'Algebraicity Conjecture', which states that simple groups over an algebraically closed field are exactly the simple groups of finite Morley rank. This has been answered in the 'even type' and 'mixed type' cases, see [1], while there still seems to be a lack of methods to tackle the 'odd type' and 'degenerate type' cases. In [1], a very important tool is the classification of simple groups of finite Morley rank with a spherical Moufang *BN*-pair of Tits rank $\geqslant 2$, which was achieved in [13]. The latter makes use of the classification of Moufang polygons of finite Morley rank (also given by [13]).

The 'finite Morley rank' condition is very strong, and eliminates many interesting Moufang polygons. For example, non-principal ultraproducts of finite Moufang polygons have a very nice model

E-mail address: Pietro.DelloStritto@umons.ac.be.

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theory – they are supersimple finite rank – but do not have finite Morley rank. This inspired the work carried out in [7], which was intended to generalize the results of [13] from the superstable context (with the stronger assumption of finite Morley rank) to the supersimple one.

The work in this paper is extracted from [7]. The main results can be stated as follows (more detailed statements are, respectively, Theorems 7.2 and 8.3). First, we recall that there are, up to duality, only seven families of finite Moufang polygons, i.e. those whose members are either projective planes, symplectic quadrangles, Hermitian quadrangles in projective space of dimension 3 or 4, split Cayley and twisted triality hexagons, or Ree–Tits octagons, with the latter arising over (finite) *difference* fields (see Definition 5.15). According to Definition 3.6, every (infinite) Moufang polygon which belongs to one of those families listed above, is said to be *good*.

Theorem 1.1. *Let $\mathcal{C}$ be any of the families of finite Moufang polygons. Then $\mathcal{C}$ forms an asymptotic class.*

Theorem 1.2. *Let Γ be a good Moufang polygon, and let Σ be its associated little projective group. Then Γ and Σ are parameter bi-interpretable. In particular, Γ is supersimple finite rank if and only if Σ is supersimple finite rank.*

We also show the following result (which is Theorem 7.1), which does not seem to appear in the literature, although, to some extent, it is implicitly present in [17]; it is a fairly small extension of the results from [17]. The latter plays an important role in this paper, and we will give a quick outline of the relevant part of Ryten's thesis in Section 5.

Theorem 1.3. *For any fixed family $\mathcal{G}$ of either finite Chevalley groups or finite twisted groups of fixed Lie type and Lie rank, there exists an L_{group}-formula σ such that for any fixed finite group G, we have $G \in \mathcal{G}$ if and only if $G \models \sigma$.*

Theorem 1.1 says, essentially, that the class of definable sets in any family of finite Moufang polygons satisfies the Lang–Weil asymptotic behavior of the rational points of varieties in finite fields. The remaining work done in [7] deals with those (infinite) Moufang polygons which are not good, showing that the latter are not supersimple finite rank. This work will also be extracted and presented in a forthcoming paper, [8]; it rests on the classification of Tits and Weiss [19].

This paper is organized as follows. Sections 2 and 3 give some background on Moufang polygons (in particular, Section 3 gives examples of good Moufang polygons), while Section 5 introduces the model-theoretic notions that we will use throughout this paper; in particular, the notion for a class of finite structures to be an asymptotic class. Also, Section 4 deals with the key point regarding the interpretation of the little projective group in the polygon; this is done almost exactly as in Section 1 of [13]. Sections 6 and 7 prove Theorem 1.1. More precisely, Section 6 shows a uniform bi-interpretation (using parameters) between a given family of finite Moufang polygons and its corresponding class of finite little projective groups; since the classes of finite little projective groups are well known (they are either classes of finite Chevalley groups or finite twisted groups of fixed Lie type and Lie rank), and they are shown to be 'asymptotic classes' by [17], this uniform bi-interpretation procedure allows us to 'transfer' the asymptotic behavior to the classes of finite Moufang polygons. This is proved in Theorem 6.1. However, there is an issue regarding the use of parameters. This, in a similar context (see Chapter 5 of [17]), led Ryten to introduce the notion of a *strong* uniform bi-interpretation between classes of finite structures. This is treated in Section 7. Indeed, Theorem 7.2(i) proves that the bi-interpretation shown in the proof of Theorem 6.1 is strong. This gives Theorem 1.1. Since by [14] non-principal ultraproducts of asymptotic classes are 'measurable' and thus supersimple finite rank, Theorem 1.1 provides examples of supersimple finite rank Moufang polygons arising over (difference) pseudofinite fields.

Finally, Section 8 deals with Theorem 1.2, namely Theorem 8.2. One direction, the interpretation of the little projective group in the associated good Moufang polygon, requires just a result on the existence of a bound for the number of 'root groups' generating the little projective group, which is known to be true in the literature (see [3], for instance). For the other direction, to interpret the polygon from the group, we basically interpret the points and lines of the polygon as the coset space in the little projective group of, respectively, the stabilizer of a fixed point and the pointwise stabilizer of a fixed line, with the fixed point and line being incident.

2. Generalized polygons

In this section we introduce Moufang polygons, which are the basic objects of this paper. Moufang polygons have been classified by Tits and Weiss, and their book [19] gives full details of this classification; also, [20] gives further details on generalized polygons, including polygons without the Moufang assumption. We use both references.

Let $L_{\mathrm{inc}} = (P, L, I)$ be a language with 2 disjoint unary relations P and L and a binary relation I, where $I \subseteq P \times L \cup L \times P$ is symmetric and stands for *incidence*. An L_{inc}-structure is called an *incidence structure*. Usually, the elements a satisfying P are called *points*, those satisfying L are called *lines*, and pairs (a, l), or (l, a), satisfying I are called *flags*.

A sequence $(x_0, x_1, \ldots, x_k)$ of elements $x_i \in P \cup L$ such that x_i is incident with x_{i-1} for $i = 1, 2, \ldots, k$ is called a *k-chain*; if $x, y \in P \cup L$, and k is least such that there is a k-chain $(x_0, x_1, \ldots, x_k)$ with $x_0 = x$ and $x_k = y$, we write $\mathrm{d}(x_0, x_k) = k$. For $x \in P \cup L$, we define $B_k(x) := \{y \in P \cup L\colon\ 1 \leqslant \mathrm{d}(x, y) \leqslant k\}$. If a is a point, $B_1(a)$ is called a *line pencil*; if l is a line, $B_1(l)$ is called a *point row*.

Definition 2.1. Let $n \geqslant 3$ be an integer. A *generalized n-gon* is an incidence structure $\Gamma = (P, L, I)$ satisfying the following three axioms:

(i) every element $x \in P \cup L$ is incident with at least three other elements;
(ii) for all elements $x, y \in P \cup L$ we have $\mathrm{d}(x, y) \leqslant n$;
(iii) if $\mathrm{d}(x, y) = k < n$, there is a unique k-chain $(x_0, x_1, \ldots, x_k)$ with $x_0 = x$ and $x_k = y$.

A *subpolygon* Γ' of Γ is an incidence substructure $\Gamma' = (P', L', I') \subseteq \Gamma$, i.e. $P' \subseteq P$, $L' \subseteq L$ and $I' = I \cap (P' \times L')$, satisfying the axioms (i)–(iii) above.

Generalized n-gons are often called *thick* generalized n-gons; this is because sometimes the definition above is given with 'two' in place of 'three' in (i), and if so by dropping the assumption (i) and replacing it by:

(i)′ "every element $x \in P \cup L$ is incident with exactly two other elements",

we obtain *thin* generalized n-gons, namely *ordinary polygons*.

If confusion does not arise, we will often refer to generalized n-gons as *n-gons*, for short. For any n-gon $\Gamma = (P, L, I)$, the cardinality of a line pencil $B_1(a)$, for some $a \in P$, and the cardinality of a point row $B_1(l)$, for some $l \in L$, do not depend, respectively, on a and l; therefore, if $|B_1(a)| = s + 1$ and $|B_1(l)| = t + 1$, for some $a \in P$ and $l \in L$, where s and t can be either finite or infinite cardinals, then we define (s, t) to be the *order* of Γ. We denote by $\Gamma^{\mathrm{dual}} = (L, P, I)$ the dual of Γ, which is obtained by interchanging points and lines of Γ.

Definition 2.2. Given two incidence structures $\Gamma_1 = (P_1, L_1, I_1)$ and $\Gamma_2 = (P_2, L_2, I_2)$, an *isomorphism* of Γ_1 onto Γ_2 is a pair of bijections $\alpha : P_1 \longrightarrow P_2$ and $\beta : L_1 \longrightarrow L_2$ preserving incidence and non-incidence; a *duality* of Γ_1 onto Γ_2 is an isomorphism of Γ_1 onto Γ_2^{dual}.

Definition 2.3. Let $\Gamma = (P, L, I)$ be an n-gon. Suppose that $x, y \in P \cup L$ and $\mathrm{d}(x, y) = k < n$. By axiom (iii) of Definition 2.1, there is a unique element $z \in B_{k-1}(x) \cap B_1(y)$, which is denoted by $z = \mathrm{proj}_k(x, y)$. In particular, if $\mathrm{d}(x, y)$ is exactly n, then there is a bijection $[y, x] : B_1(x) \longrightarrow B_1(y)$, given by $z \mapsto \mathrm{proj}_{n-1}(z, y)$, with inverse $[x, y]$. We call the map $[y, x]$ a *perspectivity* between x and y; a composition of perspectivities is called a *projectivity*, and we put $[x_3, x_2][x_2, x_1] = [x_3, x_2, x_1]$, and so on.

Definition 2.4. A *root* of an n-gon Γ is an n-chain $\alpha = (x_0, x_1, \ldots, x_n)$ with $x_{i-1} \neq x_i$ for $i = 1, 2, \ldots, n$. Given such a root α, consider the set $X = \bigcup_{i=1}^{n-1} B_1(x_i)$. We define the *root group* U_α to be the group of all automorphisms of Γ that fix X elementwise. Since U_α fixes x_0 and x_n, the root group U_α acts on both sets $B = B_1(x_0)\backslash\{x_1\}$ and $B' = B_1(x_n)\backslash\{x_{n-1}\}$. The group $\Sigma := \langle U_\alpha\colon \alpha \text{ root}\rangle$ is called the *little projective group* of the polygon Γ.

A root α is called *Moufang* if the group U_α acts transitively on the set B and, symmetrically, on the set B'; or, equivalently, on the set of all ordinary polygons containing α. Then Γ is called Moufang if every root α is Moufang.

There are basically two ways of coordinatizing a generalized polygon. We follow a purely geometric approach as in [13] and [20], while the Tits and Weiss classification follows a more algebraic path.

Definition 2.5. Let u, v be a flag of an n-gon Γ. Then, for some $k < n$, we define $B_k(u, v) = B_k(v)\backslash B_{k-1}(u)$ to be a *Schubert cell* of Γ. In particular, since for some flag (a, l) we have that $P = B_0(l, a) \cup B_1(a, l) \cup B_2(l, a) \cup \cdots$, the set of points P is partitioned into n Schubert cells. Likewise for the set of lines L.

Definition 2.6. Let $\Gamma = (P, L, I)$ be a generalized n-gon, and let $A = (x_0, x_1, \ldots, x_{2n-1})$ be an ordinary polygon in Γ. Consider an element $x \in B_k(x_{2n-1}, x_0)$, for some $k < n$, and let $(x_{2n-1}, x_0, x'_1, x'_2, \ldots, x'_k = x)$ denote the corresponding $(k+1)$-chain. Note that $\mathrm{d}(x'_i, x_{n+i}) = n$, for $i = 1, 2, \ldots, k$, so we may put $t_i(x) = \mathrm{proj}_{n-1}(x'_i, x_{n+i-1}) \in T_i$, where $T_i = B_1(x_{n+i-1})\backslash\{x_{n+i}\}$ are the *parameter sets*. We have therefore attached *coordinates* $(t_1(x), t_2(x), \ldots, t_k(x)) \in T_1 \times T_2 \times \cdots \times T_k$ to the element x.

Above, we considered only elements at distance k from x_0 which are not at distance $k-1$ from x_{2n-1}. Thus, we can attach coordinates to the remaining elements treating them as elements of the Schubert cells $B_k(x_0, x_{2n-1})$; for example, if $x \in B_k(x_0, x_{2n-1})$, for $k \leqslant n-1$, then the first element x'_1 of the $(k+1)$-chain joining x with the flag (x_0, x_{2n-1}) is now *opposite* to (i.e. at distance n) x_{n-2}, and not to x_{n-1} as in the previous case; thus, the coordinates of x with respect to the Schubert cell $B_k(x_0, x_{2n-1})$ are $t_i(x) = \mathrm{proj}_{n-1}(x'_i, x_{n-i}) \in T_{n-1+i}$ for $i = 1, 2, \ldots, k$, where $T_{n-1+i} = B_1(x_{n-i})\backslash\{x_{n-i-1}\}$.

It follows that the coordinatization uses $2n-2$ *parameter sets*, namely the sets $T_1, T_2, \ldots, T_{n-1}$ for the Schubert cells $B_k(x_{2n-1}, x_0)$ with $k = 1, 2, \ldots, n-1$, and the sets $T_n, T_{n+1}, \ldots, T_{2n-2}$ for the Schubert cells $B_k(x_0, x_{2n-1})$ with $k = n, n+1, \ldots, 2n-2$.

Remark 2.7. Let Γ, A and T_i, be as in Definition 2.6. We call the set $X = \bigcup_{i=0}^{2n-1} B_1(x_i)$ the *hat-rack* of Γ. Since from the coordinatization every element $x \in P \cup L$ has coordinates from the parameter sets T_i, it follows that, model theoretically, $\mathrm{dcl}(X) = \Gamma$ (see fourth paragraph at the beginning of Section 5).

Remark 2.8. Typically, there is an algebraic structure S (i.e. an alternative division ring, a vector space over a field, a Jordan division algebra, and so on), two subsets S_1 and S_2 of S, and functions from $S_1 \times S_1$, $S_1 \times S_2$ and/or $S_2 \times S_1$ to S_1 and/or S_2 (for example, a bilinear form, a quadratic form, a norm map, and so on), which 'determine', up to duality, the associated generalized polygon, and vice versa. For instance, sometimes S_1 has the structure of a field, S_2 that of a vector space over S_1, and the map $S_2 \longrightarrow S_1$ is a quadratic form (this is the case of an orthogonal quadrangle – see Example 3.2).

In this paper, we will sometimes use the following informal meaning of coordinatization: given a generalized polygon Γ, we say that Γ is *coordinatized by*, or *coordinatized over*, the structure S, if S is the algebraic structure associated to Γ as meant in the last paragraph; see Part II, Sections 9–16, and Part III, Section 30, of [19], for all the details about these algebraic structures. This is not used in a precise model-theoretic sense.

Generalized polygons are interesting objects in their own right, but they have particular significance because of the strong connection with the theory of Tits buildings; for background on buildings, see any of [2,3,16,18,19]. Buildings were introduced by Tits as an attempt to give a geometric interpretation of the semi-simple Lie groups and, in particular, the exceptional groups. Roughly, a building is a simplicial complex which is glued together, in a certain regular fashion, from multiple copies of a family of subcomplexes called apartments, and it is subject to a few axioms; one of the latter says that every chamber (i.e. a maximal simplex) is contained in some apartment. Each building has an associated Coxeter group W, which determines the structure of the apartments.

Definition 2.9. A *Coxeter group* is a group W generated by a set of involutions I with the following presentation:

$$W = \langle I:\ i^2 = 1 \text{ and } (ij)^{m_{ij}} = 1 \text{ for all distinct } i, j \in I \rangle.$$

The *Coxeter matrix* of W is the $n \times n$ symmetric matrix with entries m_{ij}.

A building is said to be spherical (irreducible) if its associated Coxeter group W is finite (irreducible), and it is said to have rank r, for some $r \in \mathbb{N}$, if the cardinality of the index set I of W is r. If G is a group which acts on a building as a type-preserving automorphism group, then it is called *strongly transitive* if it acts transitively on the pairs (A, C), where A is an apartment and C is a chamber in it. Given such a pair (A, C), if we let $B = G_C$ denote the stabilizer of C in B, and let N denote the group of all elements of G fixing A setwise, then the pair (B, N) forms a BN-pair of G (see, for instance, Definition 33.2 of [19]), and the group $W = N/(N \cap B)$ may be identified with the Coxeter group of the building. With this proviso, the *parabolic subgroups* of G are the stabilizers G_X as X varies through the simplexes in C, and they are precisely the subgroups of G which contain the subgroup B, also called *Borel subgroup*. For instance, if $G = \mathrm{SL}_n(\mathbb{C})$, the group of invertible $n \times n$ complex matrices, then B is the subgroup of upper triangular matrices, $T = B \cap N$ that of diagonal matrices (also called *torus*), and N that of permutation matrices (i.e. those having exactly one entry 1 in each row and each column, and 0's elsewhere), with N being the normalizer of T in G.

In the particular case of a spherical, irreducible building of rank 2, namely a Moufang generalized n-gon, a pair (A, C) is nothing but a pair of an ordinary subpolygon A containing a point-line flag $C = pIl$, and the corresponding parabolic subgroups are the stabilizers G_p and G_l of G.

We end this section with some important results in the theory of buildings and generalized n-gons.

Theorem 2.10. *Let $(\Delta, \mathcal{A})$ be an irreducible, spherical building of Tits rank $\geqslant 3$, with associated Coxeter matrix $M = (m_{ij})_{i,j \in I}$. Then:*

(i) *every residue of rank 2 is a generalized m_{ij}-gon for some $i, j \in I$ (see Proposition 3.2 of [16]);*
(ii) *rank 2 residues have the Moufang property.*

The final theorem that we mention is a generalization of an earlier result of W. Feit and G. Higman on thick finite generalized n-gons, where the latter were shown to exist only for $n = 3, 4, 6$ and 8. If the order of a generalized n-gon Γ is (s, t), and if s and t are both infinite, then Γ can exist for each $n \geqslant 2$. However, as in the finite case, which provides a bound for n, Tits and Weiss (see Theorem 17.1 of [19]) obtained a similar result in the infinite case, provided that the generalized n-gon is Moufang.

Theorem 2.11. *Moufang n-gons exist only for $n \in \{3, 4, 6, 8\}$.*

3. Some examples of Moufang polygons

We now give an introduction to certain families of (Moufang) generalized polygons from the Tits and Weiss classification which, up to some restriction, also arise in the finite case. Throughout this paper, by a *skew-field* we mean a non-commutative division ring.

Example 3.1. *Triangles* ($n = 3$): Generalized 3-gons are precisely projective planes. By Theorem 17.2 of [19], Moufang projective planes are coordinatized by alternative division rings; see also Construction 2.2.4 of [20]. Alternative division rings are either associative (fields or skew-fields) or non-associative (Cayley–Dickson algebras, see Definition 9.11 of [19]).

We denote by $\mathrm{PG}_2(A)$ the Moufang projective plane coordinatized by an alternative division ring A. Any finite Moufang projective plane is Desarguesian over a finite field, and we denote it by $\mathrm{PG}_2(q)$ for some finite field $\mathbb{F}_q$ with q a prime power.

Example 3.2. *Orthogonal and Hermitian quadrangles* ($n = 4$): Let V be a vector space over some (skew) field K, and let σ be a field anti-automorphism of order at most 2, i.e. $(ab)^\sigma = b^\sigma a^\sigma$, for all $a, b \in K$. Put $K_\sigma = \{t^\sigma - t\colon\ t \in K\}$. Then consider a *$\sigma$-quadratic form* $q : V \longrightarrow K/K_\sigma$ such that $q(x) = g(x, x) + K_\sigma$, for all $x \in V$, where g is the '$(\sigma, 1)$-linear form' associated to q; see Section 2.3 of [20] for the details.

We say that q has *Witt index l*, for some $l \in \mathbb{N}$, if $q^{-1}(0)$ contains l-dimensional subspaces but no higher dimensional ones. For a non-degenerate σ-quadratic form q on K with Witt index 2, we define the following geometry $\Gamma = Q(V, q)$: the points are the 1-spaces in $q^{-1}(0)$, the lines are the 2-spaces in $q^{-1}(0)$, and incidence is symmetrized inclusion. By Corollary 2.3.6 of [20], Γ is a generalized quadrangle if and only if V has dimension $\geqslant 5$ or $\sigma \neq \mathrm{id}_K$ (and $\dim V \geqslant 4$). All such quadrangles with σ being the id_K are called *orthogonal quadrangles*. We denote them by $Q(l, K)$, for $l := \dim(V) \geqslant 5$. The remaining ones, where $\sigma \neq \mathrm{id}_K$, give rise to *Hermitian quadrangles*, which are constructed over vector spaces of dimension $l \geqslant 4$; we denote them by $HQ(l, K)$.

For any finite field $\mathbb{F}_q$, with q a prime power, orthogonal quadrangles arise only over a vector space of dimension 5 or 6 (see Section 2.3.12 of [20]), and we denote them by $Q(5, q)$ and $Q(6, q)$, respectively. Likewise, over some finite field $\mathbb{F}_q$, there are only two examples (up to duality) of Hermitian quadrangles, and we denote them by $HQ(4, q)$ and $HQ(5, q)$ (see again Section 2.3.12 of [20]).

Remark 3.3. The orthogonal quadrangle $Q(5, K)$ is sometimes seen, under the Klein correspondence, as a quadrangle isomorphic to the dual (see Definition 2.2) of the symplectic quadrangle $W(K)$, say, over the same field (see Proposition 3.4.13 of [20] for the details). The *symplectic quadrangle* $W(K)$ consists of the totally isotropic 1 and 2-subspaces of the 4-dimensional vector space equipped with a non-degenerate symplectic form, and incidence is given by inclusion.

There are also two examples of anti-isomorphisms between orthogonal and Hermitian quadrangles: $HQ(4, L)$ is isomorphic to the dual of the orthogonal quadrangle $Q(6, K)$, for some quadratic Galois extension L of the field K equipped with a non-trivial element $\sigma \in \mathrm{Gal}(L/K)$ (see Proposition 3.4.9 of [20]); $HQ(5, L)$ is dual to the orthogonal quadrangle $Q(8, K)$, where L is a skew-field which is a quaternion algebra of dimension 4 over its centre K (see Proposition 3.4.11 of [20]).

Example 3.4. *Split Cayley and twisted triality hexagons* ($n = 6$): Let V be an 8-dimensional vector space over a field K, and let $\mathcal{Q}_7(K)$ be the non-degenerate *quadric hypersurface* of Witt index 4 living in the associated 7-dimensional projective space $P(V)$ of V; see Section 2.4 of [19] for the details. The hexagons we are interested in arise from the quadric $\mathcal{Q}_7(K)$. By the Witt index 4 assumption, $\mathcal{Q}_7(K)$ contains 3-dimensional projective subspaces of $P(V)$.

With regards to the quadric $\mathcal{Q}_7(K)$, there exists a certain 'trilinear form' $T : V \times V \times V \longrightarrow K$ (see Section 2.4.6 of [20]) such that, for some fixed $v \in V\backslash\{0\}$, the set of all $w \in V$ for which $T(v, w, x)$ vanishes in x is a projective 3-space of $\mathcal{Q}_7(K)$; moreover, the vanishing of $T(v, w, x)$ provides an incidence structure whose points are such projective 3-spaces, in a way that it also allows us to represent these points as points of $P(V)$. Then this arising point-line incidence structure (where the lines are just the lines of $P(V)$) turns out to be a generalized hexagon; see Theorem 2.4.8 of [20]. There are two kinds of hexagons, and they both depend on a certain automorphism σ of K of order 1 or 3. If $\sigma = \mathrm{id}_K$, we call the associated hexagon a *split Cayley hexagon*, and denote it by $H(K)$, and if $\sigma \neq \mathrm{id}_K$, we call it a *twisted triality hexagon*, and denote it by $T(K, K^\sigma)$.

In the finite case, the field automorphism σ is determined by the field $\mathbb{F}_{q^3}$, where q is a prime power, and therefore the finite twisted triality hexagon is unique, namely $T(q^3, q)$.

Example 3.5. *Ree–Tits octagons* ($n = 8$): These are associated to a certain *metasymplectic space* M, which is a building arising from a Dynkin diagram of type F_4; see Theorem 2.5.2 of [20]. Associated to M there is a field K of characteristic 2 which admits a Tits endomorphism σ (see Definition 5.16). The generalized octagons associated to the pair (K,σ) are called *Ree–Tits octagons*, and denoted by $O(K,\sigma)$.

Moufang octagons arise over fields of order 2^{2k+1}, and in this case the Tits endomorphism is always the automorphism $x \longmapsto x^{2^k}$ (see Lemma 7.6.1 of [20]). Thus, we denote a finite Ree–Tits octagon by $O(2^{2k+1}, x \longmapsto x^{2^k})$.

Definition 3.6. With the notation of Examples 3.1, 3.2, 3.4 and 3.5, we call *good polygons* all the following generalized polygons: $\mathrm{PG}_2(A)$ over a Desarguesian division ring A, $Q(l,K)$ for $l = 5, 6$, $HQ(l,K)$ for $l = 4, 5$, $H(K)$, $T(K,K^\sigma)$ and $O(K,\sigma)$, with K a perfect field.

4. Definability of the root groups

With the notation of Section 2, let us fix a (Moufang) generalized n-gon Γ, an ordinary subpolygon $A = (x_0, x_1, \dots, x_{2n-1})$ in Γ, a root $\alpha = (x_0, x_1, \dots, x_n) \subseteq A$, and the root group U_α associated to α. We now discuss the procedure which allows us to define U_α in the language L_{inc}; this is extracted from [13]. For the model-theoretic concept of definability see the fourth paragraph at the beginning of Section 5.

Put $B = B_1(x_{2n-1}, x_0) = B_1(x_0)\backslash\{x_{2n-1}\}$ and $0 = x_1$. Next, choose an element $a \in B_1(x_{2n-1})\backslash\{x_0, x_{2n-2}\}$. For $y \in B$, put $a_y = \mathrm{proj}_{n-1}(a, [x_n, x_0](y))$ and consider the projectivity $\pi_y = [x_0, a_y, x_{2n-2}, a_0, x_0]$ (see Definition 2.3). This projectivity fixes x_{2n-1}, induces a permutation on B, and maps 0 to y; also, π_y is parameter definable from the coordinatization (with parameters in $A \cup \{a\}$), since it is a composition of perspectivities. Hence, we have definable maps $\pm : B \times B \longrightarrow B$ by putting $x + y = \pi_y(x)$ and $x - y = \pi_y^{-1}(x)$. The structure $(B, +)$ is a *right* (or *left*) *loop*, i.e. satisfies the following: $(x + y) - y = (x - y) + y = x$, $0 + x = x + 0 = x - 0 = x$.

Let now g be an element of U_α, and put $c := g(0)$. Then, since g is an automorphism, $g(a_0) = g(a_c)$. By Lemma 1.13 of [13], it follows that $g(x) = [x_0, a_c, x_{2n-2}, a_0, x_0](x) = x + c$ for all $x \in B$ and, similarly, that $g^{-1}(x) = x - c$. In particular, the lemma tells us that given any element $g \in U_\alpha$, the restriction of g to B, denoted $g|_B$, is definable with parameters in $A \cup \{a\}$. Also, from Lemmas 1.15 and 1.16 of [13], it follows, respectively, that the action of U_α on B is *semi-regular*, i.e. 1_{U_α} is the only element in U_α fixing any element of B, and that U_α is embedded into $(B, +)$ via the map $g \longmapsto g^{-1}(0)$.

Assuming now the Moufang condition, U_α acts transitively (and thus regularly, by the above semi-regularity) on the set B; therefore, we can definably identify the root group U_α with the additive loop $(B, +)$. In order to obtain a definable action of U_α on Γ, we need to definably extend the action of $(B, +)$ to the whole of Γ. This can be done using the coordinatization as in Definition 2.6; for an important role is played by the Beth's Definability Theorem (see, for instance, Theorem 5.5.4 of [11]). The latter is a theorem in mathematical logic which deals with a common issue in mathematics, namely that of questioning to what extent implicit definitions can be made explicit. Informally, it says that for every first-order theory T in a given language, if the addition of a single relation symbol to the language of T suffices to define a property uniformly on every model of T, then this property is in fact already first-order definable using the language of T.

In the following lemma, we can use Beth's Definability Theorem since the action of each element g of U_α on the whole of Γ is implicitly definable in a language expansion L'_{inc} of L_{inc} whose models have all the same reduction to L_{inc}.

Lemma 4.1. *The action of U_α on Γ is parameter definable in L_{inc}.*

Proof. Let Γ, A, α, and so on, be as in the above setting. We have shown how to definably identify (using parameters from $A \cup \{a\}$) the root group U_α with the right loop $(B, +)$. Hence, for the assertion, we need to show that the action of every element $g \in U_\alpha$ definably extends to the whole of Γ.

Let g be any element of U_α, and consider $g|_B$. Let $X = B \cup B_1(x_1) \cup B_1(x_2) \cup \cdots \cup B_1(x_{n-1})$. Then $g|_X$ has a unique extension to an automorphism of Γ: indeed, the action of $g|_X$ on the ordinary polygon A is uniquely determined; using perspectivities the action on the corresponding hat-rack is determined, and hence, by Remark 2.7, the action of g on Γ is uniquely determined.

Consider now the language $L = (P, L, I, c_0, c_1, \ldots, c_n, Q, g|_Q)$, where the c_i are constant symbols, and Q and $g|_Q$ are relation symbols of arity 1 and 2, respectively. Let T be the first-order L-theory describing the above structure with the constant symbols c_i interpreted as the elements x_i of α, and Q interpreted as $B \cup B_1(x_1) \cup B_1(x_2) \cup \cdots \cup B_1(x_{n-1})$. Let also $L^+ = L \cup \{g\}$, where g is a binary relation symbol. Let T^+ be the L^+-theory extending T which asserts that g is the graph of an automorphism of $\Gamma = (P, L, I)$ extending $g|_X$. By the last paragraph, any model of T has a unique extension to a model of T^+. Hence, by Beth's Definability Theorem, g is uniformly definable in models of T. $\square$

We close this section with a very important tool in the study of groups of finite Morley rank (the latter is a model-theoretic notion of dimension which generalizes that of an algebraic variety), namely the Zilber Indecomposibility Theorem. The latter originally appeared in [23] in a more general setting, but we state it in the viewpoint of [1]. First, a definable subset A of a definable group G is said to be *indecomposable* if for every definable subgroup $H < G$, either A lies in a single coset of H, or A intersects infinitely many cosets of H non-trivially.

Theorem 4.2. *Let G be a group of finite Morley rank. Let $\{A_i\colon i \in I\}$ be indecomposable subsets of G, each containing the identity. Then $H := \langle A_i\colon i \in I\rangle$ is definable, connected and equals a finite product $A_{i_1} A_{i_2} \ldots A_{i_m}$.*

However, since we are interested in the supersimple finite rank context, we now state the theorem in a more general version valid for supersimple theories; for the latter see Definition 5.5(3). It draws on an earlier version of the Zilber Indecomposibility Theorem given by Hrushovski (Theorem 7.1 of [12]) in the context of so-called S_1-theories. Later on, Wagner reformulated it in the general setting of supersimplicity of finite rank as Theorem 5.5.4 of [21]; this has a very general assumption on the type-definability of the group. We take Ryten's point of view and state the theorem as it appears in a paper of Elwes and Ryten, Remark 3.5 of [10]; here, and elsewhere, we use the following notation: if H is a subgroup of a group G, and $X \subseteq G$, then $X/H := \{xH\colon x \in X\}$.

Theorem 4.3. *Let G be a group definable in a supersimple structure of finite rank. Let $\{X_i\colon i \in I\}$ be a collection of definable subsets of G. Then there exists a definable subgroup H of G such that*:

(i) *$H \subseteq \langle X_i\colon i \in I\rangle$ and every element of H is a product of a bounded finite number of elements of the X_i's and their inverses; in fact, there are $i_1, i_2, \ldots, i_n \in I$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{\pm\}$ such that $H \leqslant X_{i_1}^{\epsilon_1} X_{i_2}^{\epsilon_2} \ldots X_{i_n}^{\epsilon_n}$*;
(ii) *X_i/H is finite for each $i \in I$.*

5. Asymptotic classes of finite structures

We assume that the reader is familiar with the basic notions of model theory; in particular, the concepts of a first order language L and an L-structure M, that is, a structure interpreting L. Usually, A, B, etc., will denote subsets of M, and x, y, etc., will denote elements of M. Unless it is clear from the context, $\bar{x}$ will denote a tuple $(x_1, x_2, \ldots, x_n) \in M^n$, for some integer n. If $\bar{b} = (b_1, b_2, \ldots, b_n) \in B^n$, we often abuse notation by writing $\bar{b} \subseteq B$. L-formulas will be denoted by Greek letters ϕ, ψ, θ, etc., and we will distinguish between their variables, usually denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$, etc., and *constant variables* or *parameters*, usually denoted by $\bar{a}, \bar{b}, \bar{c}$, etc. If we have an L-structure M and a subset $A \subseteq M$, by $L(A)$ we mean the language obtained by expanding that of L with new constant symbols, one for each element of A.

An L-formula $\phi(\bar{x}, \bar{a})$ is said to be an *L-sentence* if the variables $\bar{x}$ appear only under the existential quantifier $\exists$ and/or the universal quantifier $\forall$. In general T will denote a complete theory in the language L, where by 'complete' we mean that it is a maximal consistent set of L-sentences. For an L-formula $\phi(\bar{x}, \bar{a})$, with parameters in an L-structure M, by the expression $M \models \sigma$ we mean that σ

is true in M. If N is another L-structure, then we say that N is an *elementary substructure of* M if for every L-formula $\phi(\bar{x}, \bar{a})$ with parameters in N that has a solution in M also has a solution in N when evaluated in M. If N is an elementary substructure of M, then we say that M is an *elementary extension* of N.

Given an L-structure M, by a *partial type over* A we mean a set of L-formulas with parameters from $A \subseteq M$, which is realized in some elementary extension $\bar{M}$ of M; while for a *complete type over* A, denoted by $\mathrm{tp}(\bar{x}/A)$, we mean a partial type which contains either ϕ or $\neg\phi$ for every L-formula ϕ with parameters in A. Moreover, we say that an L-structure M is κ*-saturated*, for some cardinal κ, if every type over a subset of cardinality less than κ is realized in M; in particular, M is *saturated* if it is $|M|$-saturated. Also, M is said to be κ*-homogeneous* if any partial automorphism between two subsets of cardinality less than κ can be extended to an automorphism of M. A key point is that every complete theory T in a first-order language L has a κ-saturated and κ-homogeneous L-structure M, for all cardinals κ (see, for instance, Lemma 2.1.1 of [21]); such an L-structure is commonly known as the *monster model* of the theory T, and we will denote it by $\bar{M}$. Given an L-structure M, we often refer to $\mathrm{Th}(M)$ as the *theory of* M, i.e. the theory consisting of those first order sentences true in M.

A set $D \subseteq M^n$ is said to be *definable*, over $B \subseteq M$, if there is some L-formula $\phi(\bar{x}, \bar{b})$, with parameters $\bar{b} \subseteq B$, such that $\phi(\bar{x}, \bar{b})$ is satisfied exactly by elements of D. Sometimes we will denote it by $D = \phi(M^n, \bar{b})$. When we define a set over the *empty set*, we talk about a 0-definable set. If D is a finite B-definable set $\{a_1, a_2, \dots, a_n\}$, the elements a_i are said to be *algebraic* over B; in particular, if A is a singleton $\{a\}$, then a is said to be in the *definable closure* of B, which is denoted by $\mathrm{dcl}(B)$. An *interpretable set* is a set of the form A/E, where $A \subseteq M^n$ is a definable set and E a definable n-ary equivalence relation on A.

We now move to the concept of uniform definability for infinite families of finite structures. Let L be a first-order language, and let M be an L-structure. Also, let $\mathcal{C}$ be a family of finite L-structures and $\phi(\bar{x}, \bar{y})$ be an L-formula in the variables $\bar{x}$ and $\bar{y}$. Clearly, by plugging parameters $\bar{a} \in M^{l(\bar{y})}$, for some $M \in \mathcal{C}$, into the formula $\phi(\bar{x}, \bar{y})$ so that $M \models \exists x(\phi(\bar{x}, \bar{a}))$, we obtain a family $\{\phi(\bar{x}, \bar{a})\}_{\bar{a} \in M^{l(\bar{y})}}$ of parameter definable subsets of M; the set of all such parameters $\bar{a}$ will be called *parameter set of* $\phi(\bar{x}, \bar{y})$ *in* M, and denoted by $P(\phi(\bar{x}, \bar{y}))(M)$. A *parameter definable family of* $\phi(\bar{x}, \bar{y})$ *in* M is then given as $\phi(\bar{x}, \bar{y}) \wedge Q(\bar{y})$, where $Q(\bar{y})$ is an $L(M)$-formula and $Q(M) \subseteq P(\phi(\bar{x}, \bar{y}))(M)$; if no parameters are involved in the formula $Q(\bar{y})$, then we say that the family is 0-definable. By a *stratification of* $\phi(\bar{x}, \bar{y})$ *in* M, we mean a partition $S = \{\phi(\bar{x}, \bar{y}) \wedge Q_j(\bar{y})\colon\ j \in J\}$, for some index set J, of mutually exclusive (parameter) definable families of $\phi(\bar{x}, \bar{y})$ in M so that $P(\phi(\bar{x}, \bar{y}))(M) = \bigcup_{j \in J} Q_J(M)$. If we now consider the whole of $\mathcal{C}$ and 0-definable formulas $\{Q_j(\bar{y})\colon\ j \in J\}$ which stratify $\phi(\bar{x}, \bar{y})$ in M, for every $M \in \mathcal{C}$, then we say that the stratification is *uniformly definable for* $\mathcal{C}$. In particular, each $Q_j(\bar{x}, \bar{y})$ is a uniformly definable sub-family (with respect to $\phi(\bar{x}, \bar{y})$) for $\mathcal{C}$.

We focus on a model-theoretic generalization of results on finite fields in [5], stemming ultimately from Lang–Weil. It was introduced (in dimension 1) in [14], and extended to arbitrary finite dimension by Elwes in [9].

Definition 5.1. (See Definition 2.1 of [9].) Let L be a countable first order language, $N \in \omega$, and $\mathcal{C}$ a class of finite L-structures. Then we say that $\mathcal{C}$ is an *N-dimensional asymptotic class* if for every L-formula $\phi(\bar{x}, \bar{y})$, where $l(\bar{x}) = n$ and $l(\bar{y}) = m$, there is a finite set of pairs $D \subseteq (\{0, 1, \dots, Nn\} \times \mathbb{R}^+) \cup \{(0, 0)\}$ and for each $(d, \mu) \in D$ a collection $\Phi_{(d,\mu)}$ of elements of the form $(M, \bar{a})$, where $M \in \mathcal{C}$ and $\bar{a} \in M^m$, so that $\{\Phi_{(d,\mu)}\colon\ (d, \mu) \in D\}$ is a partition of $\{\{M\} \times M^m\colon\ M \in \mathcal{C}\}$, and $|\phi(M, \bar{a})| - \mu|M|^{d/N} = o(|M|^{d/N})$, as $|M| \longrightarrow \infty$ and $(M, \bar{a}) \in \Phi_{(d,\mu)}$. Moreover, each $\Phi_{(d,\mu)}$ is required to be definable, that is to say $\{\bar{a} \in M^m\colon\ (M, \bar{a}) \in \Phi_{(d,\mu)}\}$ is uniformly 0-definable across $\mathcal{C}$.

Remark 5.2. In order to check that $\mathcal{C}$ is an N-dimensional asymptotic class, it suffices to verify the above conditions for all formulas $\phi(\bar{x}, \bar{y})$ where $l(\bar{x}) = 1$; see Lemma 2.2 of [9].

Definition 5.3. Let $\mathcal{C}$ be a class of finite L-structures, and let N be a positive integer. We say that $\mathcal{C}$ is a *weak N-dimensional asymptotic class* if it satisfies the asymptotic behavior as in Definition 5.1 but without the assumption that $\Phi_{(d,\mu)}$ is definable. Also, we say that $\mathcal{C}$ is a *semiweak N-dimensional asymptotic class* if $\Phi_{(d,\mu)}$ is uniformly definable but not necessarily 0-definable.

As examples of 1-dimensional asymptotic classes we mention: finite fields; for every finite $d \geqslant 2$, the class of all finite vertex transitive graphs of valency d; finite extraspecial groups of exponent a fixed odd prime number p; finite cyclic groups. See [14] for the details about these examples. By [6], any smoothly approximable structure is approximated by a sequence of 'envelopes'; a carefully chosen class of finite envelopes forms an N-dimensional asymptotic class. As an example, we mention, over a fixed finite field $\mathbb{F}_q$, the class of all finite dimensional vector spaces equipped with a non-degenerate alternating bilinear form.

An important model-theoretic concept in this paper is that of supersimplicity (or more strictly that of measurability) by which we mean first-order theories that are subclasses of simple theories equipped with a rank on types. The latter theories are a generalization of those which are stable, a notion introduced by Shelah. Modules and algebraically closed fields are examples of stable structures, but, for instance, vector spaces over finite fields equipped with bilinear forms, pseudofinite fields, and algebraically closed fields with a generic automorphism are examples of simple unstable theories. The key notion behind simplicity, and therefore stability, is Shelah's notion of 'dividing', or more generally 'forking'. First, we recall that a sequence $(\bar{b}_i\colon\ i \in \omega)$ of tuples is said to be *A-indiscernible*, for some set of parameters $A \subset \bar{M}$, if for every $n < \omega$ and any $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_n$, we have that $\mathrm{tp}(\bar{b}_{i_1}, \bar{b}_{i_2}, \ldots, \bar{b}_{i_n}/A) = \mathrm{tp}(\bar{b}_{j_1}, \bar{b}_{j_2}, \ldots, \bar{b}_{j_n}/A)$. Then we say that a partial type $p = p(\bar{x})$ *divides* over a set of parameters A if for some L-formula $\phi(\bar{x}, \bar{y})$, say, and some $\bar{b}$, we have: (i) $p(\bar{x}) \models \phi(\bar{x}, \bar{b})$, and (ii) there is an infinite A-indiscernible sequence $(\bar{b}_i\colon\ i \in \omega)$ such that $\bar{b}_0 = \bar{b}$, and $\{\phi(\bar{x}, \bar{b}_i)\colon\ i \in \omega\}$ is inconsistent (i.e. not realized in $\bar{M}$). Then we have the following notion of rank.

Definition 5.4. Given a formula $\phi(\bar{x})$ in a language L, with parameters contained in a set A, we define the *S_1-rank* of $\phi(\bar{x})$ as follows:

(i) $S_1(\phi(\bar{x})) = -1$ if $\phi(\bar{x})$ is inconsistent; otherwise $S_1(\phi(\bar{x})) \geqslant 0$;
(ii) for $n \geqslant 0$, $S_1(\phi(\bar{x})) \geqslant n+1$ if there is an L-formula $\psi(\bar{x}, \bar{y})$ and an A-indiscernible sequence $(\bar{c}_i\colon\ i < \omega)$ such that $\models \psi(\bar{x}, \bar{c}_i) \longrightarrow \phi(\bar{x})$ for some (any) i, and if $i \neq j$ then $S_1(\psi(\bar{x}, \bar{c}_i)) \geqslant n$ and $S_1(\psi(\bar{x}, \bar{c}_i) \wedge \psi(\bar{x}, \bar{c}_j)) < n$.

Then supersimplicity can be defined as follows:

Definition 5.5.

(1) Given three subsets $A, B, C \subset \bar{M}$ we say that *A is independent of C over B*, if $\mathrm{tp}(\bar{a}/BC)$ does not divide over C for any finite $\bar{a} \subset A$.
(2) T is *simple* if independence is a symmetric notion.
(3) T is supersimple if it is simple and for each formula $\phi(\bar{x}, \bar{y})$, it follows that $S_1(\phi(\bar{x}, \bar{y})) < \infty$.
(4) M is supersimple if $\mathrm{Th}(M)$ is.

As examples of supersimple structures, we mention pseudofinite fields and also smoothly approximable structures. Elwes proved that there is a strong connection between classes of finite structures and infinite ultraproducts arising from the members of the classes. In fact, he proved the following result, which is Lemma 4.1 of [14] generalized to N-dimensional asymptotic classes (see also Corollary 2.8 of [9]).

Proposition 5.6. *Let $\mathcal{C}$ be an N-dimensional asymptotic class and let M be an infinite ultraproduct of members of $\mathcal{C}$. Then* $\mathrm{Th}(M)$ *is supersimple and the S_1-rank of M is at most N.*

The proof of Proposition 5.6 shows that ultraproducts of members of asymptotic classes are in fact 'measurable structures' (see Definition 5.1 of [14]), which are supersimple structures of finite rank (with extra conditions) – there are corresponding notions of weak measurability and semiweak measurability. Basically, for every formula $\phi(\bar{x}, \bar{y})$ and pair (d, μ) among the finite number of possible pairs (d, μ) from Definition 5.1, there exists a corresponding formula $\phi_{d,\mu}(\bar{y})$ which assigns a dimension d and a measure μ to each set $\phi(M^n, \bar{a})$ in the ultraproduct M. This procedure is possible thanks to Los' theorem on infinite ultraproducts, which asserts that the counting arguments (on the pairs (d, μ)) hold in M since they do hold through infinitely many members of $\mathcal{C}$ but a finite number. In particular, Elwes and Ryten also proved that under these hypotheses the dimension inherited in M always exceeds the S_1-rank, which by Definition 5.5(3) implies that M is supersimple of finite S_1-rank.

We now give a quick outline of the important contribution that Ryten's thesis, [17], provides to the results which we are going to show throughout Sections 6 and 7. The content of [17] lies at the confluence of model theory, classical algebraic geometry and the theory of finite simple groups. It examines the notions of measurable groups and asymptotic classes of groups. Its main result, whose proof is deep and requires the whole thesis itself to be shown, is best represented in the following statement.

Theorem 5.7. *Any family of finite simple groups of Lie type of bounded Lie rank forms an asymptotic class.*

We now introduce the concept of bi-interpretation, i.e. the interpretation of a structure into another, and vice versa, which plays an important role in this paper. Bi-interpretation can be formulated as a concept between *classes* of finite, or infinite, structures, and it is a key tool for the results achieved in [17].

Definition 5.8. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of structures in first order languages L_1 and L_2, respectively. We say that $\mathcal{C}_1$ is *uniformly parameter interpretable*, UPI, in $\mathcal{C}_2$ if there exists an injection $i : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ so that for each $M \in \mathcal{C}_1$, the L_1-structure M is (parameter) interpreted in $i(M)$, uniformly across $\mathcal{C}_1$, i.e. there exists an L_2-formula $\phi(\bar{u}, \bar{z})$ such that for every $M \in \mathcal{C}_1$ there are $r = l(\bar{u})$, a definable set $X = \phi(\bar{u}, \bar{a}) \subset i(M)^r$ for some tuple $\bar{a}$ of $i(M)$ of the same length as $\bar{z}$, an L_2-definable equivalence relation $E = E(\bar{u}_1, \bar{u}_2)$ (defined over $\bar{a}$) on X with $l(\bar{u}_1) = l(\bar{u}_2) = l(\bar{u})$, a map $f_{\mathcal{C}_1} : M \longrightarrow X/E$, and L_2-definable subsets (defined over $\bar{a}$) of the Cartesian powers of X/E which interpret the constant, relation, and function symbols of L_1 in such a way that $f_{\mathcal{C}_1}$ is an L_1-isomorphism. We call $M^\star$ the interpretation of M in $i(M)$, and denote by $f : M \longrightarrow M^\star$ the associated L_1-isomorphism. If $\bar{a}_z$, say, is the tuple of $i(M)$, or an 'imaginary' tuple of X/E, that is used as parameters to interpret $M^\star$, then we call $\bar{a}_z$ the *witness* to the UPI in $\mathcal{C}_2$.

Suppose now that the map i is a bijection, and that $\mathcal{C}_2$ is also UPI in $\mathcal{C}_1$ (i.e. there exists an L_1-formula $\psi(\bar{x}, \bar{y})$ such that for every $N \in \mathcal{C}_2$ there are s, $Y = \psi(\bar{x}, \bar{a}_y) \subset i^{-1}(N)^s$, E', $g_{\mathcal{C}_2} : N \longrightarrow Y/E'$, etc., as before). Thus, denote by $g : N \longrightarrow N^\star$ the L_2-isomorphism associated with the interpretation of N in $i^{-1}(N)$, for every $N \in \mathcal{C}_2$. Then g induces an L_1-isomorphism $g^\star : M^\star \longrightarrow M^{\star\star}$, where $M^{\star\star}$ is the interpretation of $M^\star$ in $i^{-1}(M^\star)$; likewise, we have an induced L_2-isomorphism $f^\star : N^\star \longrightarrow N^{\star\star}$. With these assumptions, we say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are UPI *bi-interpretable* if the isomorphisms $g^\star f$ and $f^\star g$ are defined uniformly in the members of $\mathcal{C}_1$, and in the members of $\mathcal{C}_2$, respectively. When we say that $\bar{a}_y$ and $\bar{a}_z$ are *witnesses* to this UPI bi-interpretation, we mean, in addition to the above, that the isomorphism $g^\star f$ is $\bar{a}_y$-definable, and the isomorphism $f^\star g$ is $\bar{a}_z$-definable.

To be more precise, in [17], Ryten considers a slightly more constrained notion, which he calls a uniformly parameter definable (UPD) bi-interpretation. In the UPD case, no quotient is involved: for each $M \in \mathcal{C}$, M is parameter bi-definable with $i(M)$. Ryten proves that being an asymptotic class is preserved under *strong* uniform parameter bi-interpretability. The condition which makes a UPI bi-interpretation 'strong' was also introduced by Ryten, and we recall it in Definition 5.11 below. First, we give some motivation to this new notion.

Remark 5.9. When we have a class of finite structures $\mathcal{C}_1$ and a UPI bi-interpretation of the class $\mathcal{C}_1$ with an asymptotic class $\mathcal{C}_2$, then the asymptotic behavior of $\mathcal{C}_2$ can be 'transferred' to the class $\mathcal{C}_1$; similarly, if we have an infinite structure M parameter bi-interpretable with a measurable structure N, then (semiweak) measurability can be 'transferred' from N to M. If no parameters are involved in the bi-interpretation, then it preserves the property of being an asymptotic class (or being measurable). These results are due to Elwes and Ryten, and we mention some of them as Propositions 5.10 and 5.13.

Notice that, given an asymptotic class of finite structures $\mathcal{C}$ in a language L, and a sublanguage $L_1 \subset L$, the class of reducts $\{M_{L_1}\colon M \in \mathcal{C}\}$ may not be an asymptotic class anymore; the problem being the definability assumption required in the last clause of Definition 5.1. However, the set of reducts is a *weak* asymptotic class. On the other hand, trivially, expanding the language L by constants preserves the property of being an asymptotic class.

Proposition 5.10. *(See Corollary 3.8 of [9].) If $\mathcal{C}_1$ and $\mathcal{C}_2$, in the finite languages L_1 and L_2 respectively, are UPI bi-interpretable, and $\mathcal{C}_2$ is an asymptotic class, then there is an expansion L_1' of L_1 by finitely many constants, and for each $M \in \mathcal{C}_1$ an extension M' to L_1' so that $\mathcal{C}_1' := \{M'\colon M \in \mathcal{C}_1\}$ is an asymptotic class.*

Definition 5.11. Let $\mathcal{C}$ and $\mathcal{D}$ be two classes of finite structures, respectively, in the finite languages $L_{\mathcal{C}}$ and $L_{\mathcal{D}}$, and suppose that $\mathcal{C}$ and $\mathcal{D}$ are UPI bi-interpretable. Then they are *strongly* UPI bi-interpretable if additionally there is an $L_{\mathcal{C}}$-formula $\gamma(\bar{y},\bar{t})$ without parameters, such that if $C \in \mathcal{C}$ and $D = i(C)$ then for any $\bar{a}_y,\ \bar{a}_t \in C$, we have $C \models \gamma(\bar{a}_y,\bar{a}_t)$ if and only if $\bar{a}_y$ and $\bar{a}_z$ are witnesses to the UPI bi-interpretation between $\mathcal{C}$ and $\mathcal{D}$ (as in Definition 5.8) and $g(\bar{a}_z) = \bar{a}_t$.

Remark 5.12. Notice that the property of being strong is not in general symmetric; however, it is clear from the definition which direction we are taking, since the formula $\gamma(\bar{y},\bar{t})$ is an L-formula with L either $L_{\mathcal{C}}$ or $L_{\mathcal{D}}$. Thus, if $\gamma(\bar{y},\bar{t})$ is an $L_{\mathcal{C}}$-formula as in Definition 5.11, then we say that the UPI bi-interpretation is *strong on the $\mathcal{C}$-side*.

The following is Proposition 4.2.10(1) of [17]. Its proof requires a quite long calculation in order to 'transfer' the dimension/measure pairs from an asymptotic class to a class of finite structures. We do not give all the details, but we sketch the proof by extracting some of the main steps which highlight how this dimension/measure pairs transferring procedure works. As we remarked earlier, Ryten in [17] deals with a UPD bi-interpretation; for simplicity, this is also our point of view for the proof sketched below, and thus we are going to omit the quotients which in general might be involved in a UPI bi-interpretation. However, allowing quotients is essentially what Elwes does in Lemma 3.4 of [9], where he expands the language to a suitable one which also includes the quotients as new sorts.

Proposition 5.13. *Suppose that $\mathcal{D}$ is an asymptotic class, and $\mathcal{C}$ is strongly UPI bi-interpretable on the $\mathcal{C}$-side with $\mathcal{D}$. Then $\mathcal{C}$ is an asymptotic class.*

Sketch of the proof. Let $\psi(\bar{u},\bar{v}) \in L_{\mathcal{C}}$ be an arbitrary family of sets. Suppose $C \in \mathcal{C}$, $D \in \mathcal{D}$ and $D = i(C)$, where i is the matching of the strong UPI bi-interpretation. Let $\gamma(\bar{y},\bar{t})$ be the $L_{\mathcal{C}}$-formula involved in the strong UPI bi-interpretation as in Definition 5.11. With the notation of Definition 5.8, we also recall that there are isomorphisms $f : C \longrightarrow C^\star$ and $g : D \longrightarrow D^\star$. Consider now the 0-definable family $\psi_1(\bar{u}_1,\bar{z}\bar{v}_1)$, namely the interpretation of $\psi(\bar{u},\bar{v})$ in $L_{\mathcal{D}}$. Since $\mathcal{D}$ is an asymptotic class, the parameter set $\bar{z}\bar{v}_1$ is partitionable by formulas $\psi_{1,n_i,\mu_i}(\bar{z}\bar{v}_1)$ for $1 \leqslant i \leqslant i_0$ that give uniform asymptotic estimates across the class $\mathcal{D}$ for the cardinalities of the sub-families that they define. Likewise, we now consider the family $\psi_2(\bar{u}_2,\bar{y}\bar{t}\bar{v}_2)$ interpreted in $L_{\mathcal{C}}$, and the corresponding 0-definable sets $\psi_{2,n_i,\mu_i}(\bar{y}\bar{t}\bar{v}_2)$. We may also suppose that the family of sets that define the underlying set of C in D is given by the formula $\phi(\bar{w},\bar{z})$. Let us suppose that the latter is interpreted as $\phi_2(\bar{w}_2,\bar{y}\bar{t}) \in L_{\mathcal{C}}$, and that the parameter set $\bar{z}$ of $\phi(\bar{w},\bar{z})$ is partitionable by formulas $\phi_{n_j,\nu_j}(\bar{z})$ for $1 \leqslant j \leqslant j_0$ that give uniform asymptotic estimates across the class $\mathcal{D}$ for the cardinalities of the sub-families that they define. Likewise, we also have the corresponding 0-definable sets $\phi_{2,n_j,\nu_j}(\bar{y}\bar{t})$.

Consider now the formula $\Phi_{n_j,\nu_j}(\bar{y}\bar{t}) := \gamma(\bar{y},\bar{t}) \wedge \phi_{2,n_j,\nu_j}(\bar{y}\bar{t})$. Then for any $C \in \mathcal{C}$ and $\bar{a}_y\bar{a}_t \in C$, if $C \models \Phi_{n_j,\nu_j}(\bar{a}_y\bar{a}_t)$, it can be shown that:

$$\big||C| - \nu_j\big|i(C)\big|^{n_j/n}\big| = \circ\big(\big|i(C)\big|^{n_j/n}\big),$$

which Ryten calls ‘calibration equation’, and it tells how to rescale measure/dimension units in the $\mathcal{D}$-class to measure/dimension units in the $\mathcal{C}$-class. Likewise, recalling that the composite isomorphism $g^\star f$ is definable in the UPI bi-interpretation via a formula $i_1(\cdot, \bar{a}_y)$, say, we consider the formula $\Psi_{1,n_i,\mu_i}(\bar{y}, \bar{t}, \bar{v}_2) := \gamma(\bar{y}, \bar{t}) \wedge \bar{v}_2 = i_1(\bar{u}, \bar{y}) \wedge \psi_{2,n_i,\mu_i}(\bar{y}\bar{t}\bar{v}_2)$. Put $\Psi_{0,n_i,\mu_i}(\bar{v}) := \exists yt(\Psi_{1,n_i,\mu_i}(\bar{y}, \bar{t}, \bar{v}))$. If $C \models \Psi_{0,n_i,\mu_i}(\bar{a}_v)$, then it can be shown that:

$$\left|\left|\psi(C, \bar{a}_v)\right| - \mu_i \left|i(C)\right|^{n_i/n}\right| = \circ\left(\left|i(C)\right|^{n_i/n}\right),$$

which Ryten calls the ‘raw measure/dimension definition for a $\mathcal{C}$-set in the $\mathcal{D}$-class’. We then combine the raw measure/dimension with the calibration equation. Let $n_{ij} = n_i/n_j$ and let $\mu_{ij} = \mu_i/\nu_j^{n_{ij}}$. Consider the formula:

$$\Psi_{n_{ij},\mu_{ij}}(\bar{v}) = \exists yt\left[\Psi_{1,n_i,\mu_i}(\bar{y}, \bar{t}, \bar{v}) \wedge \Phi_{n_j,\nu_j}(\bar{y}\bar{t})\right].$$

If $C \models \Psi_{n_{ij},\mu_{ij}}(\bar{a}_v)$, then it can be shown that:

$$\left|\left|\psi(C, \bar{a}_v)\right| - \mu_{ij}|C|^{n_{ij}}\right| = \circ\left(|C|^{n_{ij}}\right),$$

and the set $\{\Psi_{n_{ij},\mu_{ij}}(\bar{v})\colon\ 1 \leqslant i \leqslant i_0,\ 1 \leqslant j \leqslant j_0\}$ stratify the family $\psi(\bar{u}, \bar{v})$ across the class $\mathcal{C}$. Thus if the dimensions of $\phi(\bar{w}, \bar{z})$ in $\mathcal{D}$ are $\{n_j\colon\ 1 \leqslant j \leqslant j_0\}$, it follows that $n_{\mathcal{C}} = \Pi_{j=1}^{j_0} n_j$ is a possible dimension for the asymptotic class $\mathcal{C}$, and that its minimal dimension is a divisor of $n_{\mathcal{C}}$. □

In Section 7 we will need the following, which is essentially Lemma 4.2.11 of [17]. The statement is adjusted here to allow UPI rather than UPD bi-interpretation.

Lemma 5.14. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are UPI bi-interpretable classes of finite structures, as above. For each $C \in \mathcal{C}$, let $\bar{a}_y$, $\bar{a}_z$ be witnesses to the bi-interpretation of C and $i(C) = D$. Suppose in addition*:

(i) *there is an $L_{\mathcal{D}}$-formula $\zeta(\bar{z})$ such that $\zeta(\bar{a}_z)$ holds, and if $\bar{a}'_z \in D$ with $\zeta(\bar{a}'_z)$, then the $L_{\mathcal{C}}$-structure whose interpretation in $L_{\mathcal{D}}$ is witnessed by $\bar{a}'_z$ is isomorphic to $C^\star$*;
(ii) *there is an $L_{\mathcal{C}}$-formula $\eta(\bar{y})$ such that $\eta(\bar{a}_y)$ holds, and if $\bar{a}'_y \in C$ with $\eta(\bar{a}'_y)$, then the $L_{\mathcal{D}}$-structure whose interpretation in $L_{\mathcal{C}}$ is witnessed by $\bar{a}'_y$ is isomorphic to $D^\star$.*

Then $\mathcal{C}$ and $\mathcal{D}$ are strongly UPI bi-interpretable, on the $\mathcal{C}$-side.

Proof. This is virtually identical to the proof of Lemma 4.2.11 of [17], except that now the interpretations allow quotients. We omit the details. □

We now give a further example of an asymptotic class which plays an important role in the content of this paper (see Remark 5.17), but we first need to introduce the following two definitions.

Definition 5.15. Let L_{diff} be the language L_{ring} augmented by a unary function symbol σ. A *difference field* is a pair (K, σ) consisting of a field K and an automorphism σ of K.

Definition 5.16. Let K be a field of characteristic p, for some prime p. A *Frobenius endomorphism* σ is the map which sends x to x^p, for every $x \in K$, and we denote it by Frob. Also, a *Tits endomorphism* of K is a square root of the Frobenius endomorphism, i.e. the endomorphism $\sigma : K \longrightarrow K$ such that $x^{\sigma^2} = \text{Frob}(x)$ for all $x \in K$.

Remark 5.17. We refer to [4] for a survey on difference fields. In [17], Ryten developed a particular theory of pseudofinite difference fields, denoted by PSF(m,n,p) (see Section 3.3.2 of [17] for the axiomatization of PSF(m,n,p)). He shows that the class of finite difference fields $\mathcal{C}_{(m,n,p)} := \{(\mathbb{F}_{p^{nk+m}}, \sigma^k)\colon k \in \omega\}$, where $m,n \in \mathbb{N}$ with $m \geqslant 1$, $n > 1$ and $(m,n) = 1$, and σ is the Frobenius automorphism, forms a 1-dimensional asymptotic class; moreover, he shows that every non-principal ultraproduct of members of $\mathcal{C}_{(m,n,p)}$ is a model of PSF(m,n,p), and, vice versa, every model of PSF(m,n,p) is elementarily equivalent to a non-principal ultraproduct of members of $\mathcal{C}_{(m,n,p)}$ (see Theorem 3.3.15 of [17]).

We close this section with a list of important results and further comments from [17], which will be used throughout Sections 6 and 7. First, Remark 5.17 suggests to rewrite Theorem 5.7 in a more accurate way as follows.

Theorem 5.18. *Let $\mathcal{G}$ be any family of finite simple Lie groups of fixed Lie rank.*

(i) *$\mathcal{G}$ is strongly UPI bi-interpretable with either the class of finite fields $\mathcal{F}$ or one of the classes $\mathcal{C}_{(m,n,p)}$;*
(ii) *$\mathcal{G}$ is an asymptotic class.*

Remark 5.19. There are just finitely many exceptions consisting of finite Moufang polygons whose corresponding little projective groups are not simple (see Lemma 5.8.1 of [20]), but we can disregard the cases in our context as Ryten does in his thesis (Remark 5.2.9 of [17]): finitely many exceptional cases can be ruled out in the bi-interpretation, by describing the elementary diagram of the corresponding models.

Proposition 5.20. *Let $\mathcal{G}$ be a class of finite Chevalley groups of a fixed Lie type and fixed Lie rank, and let $\mathcal{F}$ be the corresponding class of underlying finite* (*difference*) *fields. Assume that G comes equipped with a root system Φ and root groups X_r for each root $r \in \Phi$. For each root r there exists a homomorphism h_{SL_2} from* SL_2 *onto $\langle X_r, X_{-r}\rangle$, put $H := \langle H_r\colon r \in \Phi\rangle = \langle h_r(t)\colon h_r(t) = h_{\mathrm{SL}_2}(t,0,0,t^{-1}),\ r \in \Phi \text{ and } t \in K^\times\rangle$.*

(1) *H_r is isomorphic to $K^\times$, the multiplicative group of K, and X_r to K^+, the additive group of K.*
(2) (*See Lemma* 5.2.7 *of* [17].) *There is $m \in \mathbb{N}$ such that if $|K| > 7$, H is uniformly parameter definable across $\mathcal{G}$, as the intersection $\bigcap_{i=1}^m C_G(h_i)$ of the centralizers of some m of its non-trivial elements.*
(3) (*See Corollary* 5.2.8 *of* [17].) *The whole of H acts on X_r via conjugation, and X_r may be presented as the union of at most two H-orbits. Thus, since by the sentence above H is uniformly definable across $\mathcal{G}$, so too are the root subgroups X_r.*

6. The UPI bi-interpretation

In this section we prove Theorem 6.1, which together with Theorem 7.2 will yield Theorem 1.1. With the notation of Examples 3.1, 3.2, 3.4 and 3.5, and according to the classification of Tits and Weiss (see Section 34 of [19]), the finite Moufang generalized polygons are, up to duality, $\mathrm{PG}_2(q)$, $W(q)$, $HQ(4,q)$, $HQ(5,q)$, $H(q)$, $T(q^3,q)$ and $O(2^{2k+1}, x \mapsto x^{2^k})$.

We give the list (up to duality) of finite Moufang polygons in Table 6.1, where we associate to each polygon Γ the corresponding little projective group Σ; see, for instance, Section 8.3 of [20].

Let $\mathcal{C}$ be one of the above classes of finite Moufang polygons. To prove Theorem 1.1, we must show that $\mathcal{C}$ forms an asymptotic class (see Definition 5.1). By Proposition 5.13 (see also Proposition 5.10), in order to prove that $\mathcal{C}$ is an asymptotic class we firstly need to show that $\mathcal{C}$ is UPI bi-interpretable (see Definition 5.8) with a class $\mathcal{G}$, say, which is already known to be an asymptotic class. The 'natural candidate' in this setting is the class of corresponding finite little projective groups.

By Table 6.1, we also know that $\mathcal{G}$ is either a class of finite Chevalley groups or a class of finite twisted groups of fixed Lie type and Lie rank. Usually a little projective group $G \in \mathcal{G}$ is simple, and exceptions only occur in some small cases, i.e. when there are only three points on a line or three lines through a point of the associated Moufang polygon; more precisely, the exceptions are the

Table 6.1

Γ	Σ
$PG_2(q)$	$PSL_3(q)$
$W(q)$	$PS_{p_4}(q)$
$HQ(4,q)$	$PSU_4(q)$
$HQ(5,q)$	$PSU_5(q)$
$H(q)$	$G_2(q)$
$T(q^3,q)$	${}^3D_4(q)$
$O(2^{2k+1}, x \mapsto x^{2^k})$	${}^2F_4(q)$

groups $B_2(2)$, $G_2(2)$ and ${}^2F_4(2)$. By Theorem 5.18, $\mathcal{G}$ forms an asymptotic class which is strongly UPI bi-interpretable with a class of finite (difference) fields. Therefore, for the remainder of this section we will be exhibiting a UPI bi-interpretation between any class of finite Moufang polygons $\mathcal{C}$ and the corresponding asymptotic class of finite little projective groups $\mathcal{G}$.

Theorem 6.1. *The classes $\mathcal{C}$ and $\mathcal{G}$ are UPI bi-interpretable.*

The proof of the theorem is given throughout the rest of the section. The map $i : \mathcal{C} \longrightarrow \mathcal{G}$ is defined to take each $\Gamma \in \mathcal{C}$ to its little projective group. Since both are parametrized by a (difference) field, i is injective; in fact, by Remark 5.19 (since $\mathcal{G} \backslash i(\mathcal{C})$ is finite) we can assume, without loss of generality, that i is a bijection. We first show that $\mathcal{C}$ is UPI in $\mathcal{G}$.

Lemma 6.2. *Let $\mathcal{G}$ be a family of finite simple groups of Lie type. Then each conjugacy class of parabolic subgroups is UPI across $\mathcal{G}$.*

Proof. We first deal with the untwisted case. By Proposition 8.3.1(iii) of [3], for $\Sigma \in \mathcal{G}$ and a parabolic subgroup P_J of Σ corresponding to a set J of fundamental roots (see the discussion that follows Definition 2.9), we have $P_J = BN_JB$, so as the size of N_J does not vary over the members of $\mathcal{G}$, it suffices to show that Borel's B are uniformly definable. We also have $B = UH$, and by Proposition 5.20(2) we know that H is uniformly definable. Also, $U = X_{r_1}X_{r_2}\dots X_{r_n}$ (see, for instance, 5.3.3(ii) of [3]), for some integer n, so the assertion follows from Proposition 5.20(3).

For the twisted groups, the arguments are essentially the same. We have $B^1 = U^1H^1$ (notation of Chapter 13 of [3]), and the appropriate uniform definability results can be found in Chapter 5 of [17], which are adaptations to the twisted cases of Propositions 5.20(2) and (3). □

Lemma 6.3. *There exists a uniform parameter interpretation of the Moufang polygon $\Gamma = \Gamma(\Sigma)$ in Σ, for Σ varying through $\mathcal{G}$.*

Proof. Let Σ be the little projective group associated to a Moufang polygon $\Gamma = (P, L, I) \in \mathcal{C}$, and let pIl be a fixed flag in Γ. Denote by Σ_p and Σ_l, respectively, the stabilizer of p in Σ and the stabilizer of l in Σ. Then Σ_p and Σ_l turn out to be parabolic subgroups of the BN-pair associated to Σ; since Σ is simple, the set of pairs (Σ_p, Σ_l) such that pIl is a flag, is UPI across $\mathcal{G}$ by Lemma 6.2. Hence, from the definable parabolic subgroups of Σ we can interpret the polygon Γ as follows: interpret the points of Γ as the cosets Σ/Σ_p and the lines of Γ as the cosets Σ/Σ_l; incidence I is interpreted as $g\Sigma_p I h\Sigma_l$ if and only if $g\Sigma_p \cap h\Sigma_l \neq \emptyset$. □

In the proof of Lemma 6.2 we made use of a weaker form of the Bruhat decomposition, namely that every element of Σ can be written in the form b_1nb_2, for some $b_1, b_2 \in B$ and $n \in N$. This was enough to uniformly interpret the parabolic subgroups P_J, but we now need a refinement of such a decomposition in order to interpret the whole of Σ. This stronger decomposition represents a canonical form for elements of Σ so that each element has a unique expression in the given form; see, for instance, Corollary 8.4.4 of [3].

Proposition 6.4. *There exists a uniform parameter interpretation of the little projective group $\Sigma = \Sigma(\Gamma)$ in Γ, for Γ varying through $\mathcal{C}$.*

Proof. Let $\Gamma \in \mathcal{C}$, and let $\Sigma = \langle U_\alpha\colon \alpha \text{ is a root } \rangle$ be its little projective group. We aim to interpret Σ in a uniform way across $\mathcal{C}$. First, fix an ordinary polygon $A = (x_0, x_1, \ldots, x_{2n-1})$ in Γ, a root $\alpha = (x_0, x_1, \ldots, x_n) \subset A$ and a line pencil B centered at x_0. In Section 4 we showed how to define, with parameters, a right loop on B, how to definably identify it with the action of the root group U_α on the set B and, ultimately, how to extend such an action on the whole polygon (see Lemma 4.1); put $X = \{x_0, x_1, \ldots, x_{2n-1}, a\}$, the set of parameters used to define U_α and its action on Γ. Since Σ is generated by all its root groups, we aim to find a bound m such that for some root groups $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}$ (not necessarily distinct) $\Sigma = U_{\alpha_1} U_{\alpha_2} \ldots U_{\alpha_m}$. We follow [3].

If Σ is a Chevalley group, by the Bruhat decomposition we need to find such bounds for U, V, H and N, where U is the subgroup of Σ generated by the 'positive' root groups $U_1, U_2, \ldots, U_n$ and V that generated by the 'negative' root groups, N the group associated to the BN-pair of Σ and $H = N \cap B$. Since the set of positive roots is fixed and therefore it does not vary over the members of the fixed family of groups, U is uniformly definable as already discussed in the proof of Lemma 6.2; likewise V. Also, by Chapter 6 of [3] and the assumption of finite Lie rank r, say, every element of H is a product of $4r$ root groups; since the size of the associated Weyl group $W = N/H$ is fixed, it follows that N is also generated by a product of boundedly many root groups. For the twisted case the situation is similar as the Bruhat decomposition still holds; see Proposition 13.5.3 of [3].

Let now Σ be any finite little projective group in $\mathcal{G}$. It follows from the above paragraph that, for some integer m, we can construct Σ as a group with domain $U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_m}/{\sim}$, where the equivalence relation $\sim$ is defined as follows:

$$
\begin{aligned}
&(g_1, g_2, \ldots, g_m) \sim (h_1, h_2, \ldots, h_m) \quad \text{if and only if}\\
&g_1 g_2 \ldots g_m(x) = h_1 h_2 \ldots h_m(x), \quad \text{for } x \in P \cup L,\ g_i, h_i \in U_{\alpha_i} \text{ and } 1 \leqslant i \leqslant m.
\end{aligned}
$$

Denote by $[(g_1, g_2, \ldots, g_m)]_\sim$ the equivalence class of $(g_1, g_2, \ldots, g_m) \in U_{\alpha_1} \times \cdots \times U_{\alpha_m}$ with respect to $\sim$. Now we define the group multiplication "$\cdot$", say, as follows:

$$
\begin{aligned}
&\big[(g_1, g_2, \ldots, g_m)\big]_\sim \cdot \big[(h_1, h_2, \ldots, h_m)\big]_\sim = \big[(k_1, k_2, \ldots, k_m)\big]_\sim \quad \text{if and only if}\\
&g_1 g_2 \ldots g_m\big(h_1 h_2 \ldots h_m(x)\big) = k_1 k_2 \ldots k_m(x), \quad \text{for all } x \in P \cup L.
\end{aligned}
$$

This is clearly well defined. Without loss of generality, we may assume that U_{α_1} is U_α; then, as we defined U_α, as well as its action on the whole of the associated $\Gamma \in \mathcal{C}$ (over the set of parameters X), we can do the same for the remaining root groups U_{α_i} for $i \in 2, 3, \ldots, m$; namely, by adding new parameters $X_i = \{x_0^{(i)}, x_1^{(i)}, \ldots, x_{2n-1}^{(i)}, a^{(i)}\}$, say, for some ordinary polygons $A_i = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{2n-1}^{(i)})$ and some $a^{(i)} \in B_1(x_{2n-1}^{(i)}) \backslash \{x_0^{(i)}, x_{2n-2}^{(i)}\}$, the root groups U_{α_i} and their respective actions on the whole of Γ are definable over the set of parameters X_i for $i = 2, 3, \ldots, m$. Hence, it follows that the relation $\sim$ is definable: for any $x \in P \cup L$, $g_1 g_2 \ldots g_m(x) = h_1 h_2 \ldots h_m(x)$ if and only if the image of x under the definable action of $g_1 g_2 \ldots g_m$ is the same under the definable action of $h_1 h_2 \ldots h_m$. $\square$

Remark 6.5. The following argument can be used to find the bound m, an alternative to the method used in the proof of Proposition 6.4. Consider an infinite ultraproduct $(\Sigma^\star, U_\alpha^\star) = \Pi(\Sigma, U_\alpha)/\mathcal{U}$, for some non-principal ultrafilter $\mathcal{U}$. It follows from Proposition 5.20(3) that U_α is uniformly definable across $\mathcal{G}$ (note that in [17] the root groups are denoted by $X_r(K)$); the latter could also be shown as follows: since by Discussion 5.2.1 and, in the twisted cases, 5.3.2 and 5.4.1 of [17], the root groups $X_r(K)$ are UPD in the class $\mathcal{F}$ of the corresponding finite (difference) fields K, and since by Theorem 5.18(i) the classes $\mathcal{G}$ and $\mathcal{F}$ are strongly UPD bi-interpretable, it follows that each $X_r(K)$ is also UPD in $\mathcal{G}$. Hence, by Los' theorem on ultraproducts, the root group $U_\alpha^\star$, as its action on the whole of

$\Pi\Gamma/\mathcal{U}$, is parameter definable in $\Sigma^\star$. Also, since the cardinality $|U_\alpha|$ grows with Σ, the root group $U^\star_\alpha$ is infinite.

By Remark 5.19 and by Propositions 1, 2 and 3 of [15], $\Sigma^\star$ is a simple group, definable in the pseudofinite (difference) field $\Pi\mathbb{F}_q/\mathcal{U}$, where $\mathbb{F}_q$ denotes the underlying field of Σ as well as the underlying field of Γ.

Since $\Sigma^\star$ is generated by $\{(U^\star_\alpha)^g\colon\ g \in \Sigma^\star,\ \alpha \text{ root}\}$, and this set is $\Sigma^\star$-invariant, by the Zilber Indecomposibility Theorem (ZIT) in its supersimple finite rank version (see Theorem 4.3) there exists a definable subgroup $H \leqslant U^\star_{\alpha_1} U^\star_{\alpha_2} \dots U^\star_{\alpha_n}$ in $\Sigma^\star$, where $U^\star_{\alpha_i} := (U^\star_\alpha)^{g_i}$ for some $g_i \in \Sigma^\star$, with $i = 1, 2, \dots, n$, which is also $\Sigma^\star$-invariant, so normal; moreover, for each $i = 1, 2, \dots, n$, by ZIT we also have that $U^\star_{\alpha_i}/H$ is finite, thus $H \neq 1$. Therefore, as $\Sigma^\star$ is simple, $H = \Sigma^\star$. This argument applies to all infinite ultraproducts of the (Σ, U_α). Hence, there is a single m such that in all ultraproducts $(\Sigma^\star, U^\star_\alpha)$, we have $\Sigma^\star = U_{\alpha_1} U_{\alpha_2} \dots U_{\alpha_m}$. It follows that for all but finitely many (Σ, U_α) we have $\Sigma = U_{\alpha_1} U_{\alpha_2} \dots U_{\alpha_m}$. By increasing m to deal with the remaining finite (Σ, U_α), we may suppose that for all (Σ, U_α), we have that $\Sigma = U_{\alpha_1} U_{\alpha_2} \dots U_{\alpha_m}$ for some $\alpha_1, \alpha_2, \dots, \alpha_m$.

Lemma 6.6. *There exists a uniform parameter definable isomorphism between Γ and its re-interpretation in itself.*

Proof. Given a Moufang polygon Γ, we can re-interpret Γ in itself by first interpreting Σ in Γ as in Proposition 6.4, and then by interpreting a copy of Γ from Σ as in Lemma 6.3. This is possible because the bi-interpretation comes equipped with isomorphisms from objects to their re-interpretations, on both sides. Namely, Γ is uniformly parameter interpreted as the polygon $(\Sigma/\Sigma_p, \Sigma/\Sigma_l, \{(u\Sigma_p, u\Sigma_l)\colon\ u \in \Sigma\})$ from the group $U_{\alpha_1} \times \dots \times U_{\alpha_m}/\sim$, which is itself uniformly parameter interpreted from Γ; here the flag pIl of Lemma 6.3 is the flag $x_0 I x_{2n-1}$ fixed in Proposition 6.4. Call Γ' this re-interpretation of Γ in itself.

With the notation of Definition 5.8, we have an isomorphism $g^\star f : \Gamma \longrightarrow \Gamma'$. Put $g^\star f = \phi$. Then, by construction of $\sim$, the isomorphism ϕ is well defined; precisely, ϕ sends any point $x \in \Gamma$ (or line $l \in \Gamma$) to the unique point $y = u\Sigma_p \in \Gamma'$ (or line $y = u\Sigma_l \in \Gamma'$), with u the unique group element $u = [(u_1, u_2, \dots, u_m)]_\sim \in U_{\alpha_1} \times \dots \times U_{\alpha_m}/\sim$ such that $x = u(p)$ (or $x = u(l)$). Since by Proposition 6.4 we have a uniform parameter interpretation of the group $U_{\alpha_1} \times \dots \times U_{\alpha_m}/\sim$ and its action on the whole of Γ, we can thus uniformly define (with parameters $Y = X \cup (\bigcup_{i=2}^r X_i) \cup \{a\}$, see Proof of Proposition 6.4) the isomorphism ϕ by specifying the coset $u\Sigma_p$ such that u sends p to x.

Hence, it follows that we need the definability of the set $\{(x, u\Sigma_p)\colon\ x = u(p)\}$ in Γ. However, the latter is the following:

$$\begin{aligned}
&\bigl\{(x, u\Sigma_p)\colon\ x = u(p)\bigr\} \\
&\quad = \bigl\{\bigl(x, (u_1, u_2, \dots, u_m)/\sim \Sigma_p\bigr)\colon\ x = u_1 u_2 \dots u_m(p)\bigr\} \\
&\quad = \bigl\{(x, u_1, u_2, \dots, u_m, k_1, k_2, \dots, k_m)\colon\ x = u_1 u_2 \dots u_m(p),\ k_1 k_2 \dots k_m(p) = p\bigr\}.
\end{aligned}$$

The latter is then parameter definable in Γ, using parameters from Y. $\square$

Lemma 6.7. *There exists a uniform parameter definable isomorphism between Σ and its re-interpretation in itself.*

Proof. We start from $\Sigma \in \mathcal{G}$ and re-interpret it in itself: we first interpret (see Lemma 6.3) $\Gamma = i^{-1}(\Sigma)$ as the coset geometry $\Gamma' := (\Sigma/\Sigma_p, \Sigma/\Sigma_l, \{(u\Sigma_p, u\Sigma_l)\colon\ u \in \Sigma\})$, and then we re-interpret (see Proposition 6.4) Σ as $\Sigma' = U_{\alpha_1} \times U_{\alpha_2} \times \dots \times U_{\alpha_m}/\sim$, where pIl is a fixed flag of Γ as in Lemma 6.3.

With the notation of Definition 5.8, we have an isomorphism $f^\star g : \Sigma \longrightarrow \Sigma'$. Put $f^\star g = \psi$. Let now $u \in \Sigma$. Then, we define $\psi(u) = u'$, where for each $s\Sigma_p$ of Γ', we have $u'(s\Sigma_p) = us\Sigma_p$. Here, u' is an element in $U'_{\alpha_1} \times \dots \times U'_{\alpha_m}/\sim$, and the U'_{α_i}, for $i = 1, 2, \dots, m$, are the root groups of Γ'. Hence, we can define the set $\{(u, u')\colon\ \psi(u) = u'\}$ in Σ. $\square$

Proof of Theorem 6.1. Let $\mathcal{C}$ be any class of finite Moufang polygons, and let $\mathcal{G}$ be its associated class of finite little projective groups. Then, the UPI bi-interpretability between $\mathcal{C}$ and $\mathcal{G}$ follows immediately from Lemma 6.3 and Proposition 6.4, and also Lemmas 6.6 and 6.7. $\square$

7. Strong UPI bi-interpretation

At this stage, using Theorems 5.18 and 6.1, we know that each class of finite Moufang polygons is a semiweak asymptotic class (see Definition 5.3); i.e. we know that dimension and measure are definable, but not yet that they are 0-definable. We address this issue in this section.

The next theorem may have independent interest, but it is essentially a small extension of results from [17]. We postpone its proof until after the proof of Theorem 7.2. It will be used to verify condition (ii) of Lemma 5.14, for the UPI bi-interpretation between a class of finite Moufang polygons and the associated class of finite little projective groups. In the following, by L_{group} we mean the language of the first-order theory of groups, i.e. $L_{\text{group}} = \{\cdot, {}^{-1}, c\}$, where $\cdot$, ${}^{-1}$ and c stand for, respectively, group operation, inverse group operation and group identity symbols.

Theorem 7.1. *For any fixed family $\mathcal{G}$ of finite simple Chevalley groups, or finite twisted groups of fixed Lie type and Lie rank, there exists an L_{group}-sentence σ such that for any finite group G, we have $G \in \mathcal{G}$ if and only if $G \models \sigma$.*

Theorem 7.2.

(i) *The UPI bi-interpretation between $\mathcal{C}$ and $\mathcal{G}$ of Theorem* 6.1 *is strong, on the $\mathcal{C}$-side.*
(ii) *Each family of finite Moufang polygons forms an asymptotic class.*

Proof. First, note that (ii) follows from (i). For (i), we need to show conditions (i) and (ii) of Lemma 5.14, with $\mathcal{C}$ being a class of finite Moufang polygons and $\mathcal{D}$ the associated class of finite little projective groups, as in Theorem 6.1. To see that Lemma 5.14(i) holds, note that if $\Sigma \in \mathcal{G}$ with $\Sigma = i(\Gamma)$, then in the interpretation of Γ in Σ, the points and lines of Γ are interpreted as cosets of certain maximal parabolic subgroups. There are two cases: Γ is either a self-dual, i.e. dual of itself (see Definition 2.2), or a non-self-dual generalized polygon. Suppose first that the class $\mathcal{C}$ has self-dual members, and let the maximal parabolic subgroups P_1 and P_2, say, be defined over $\bar{a}_z$, by the formulas $\phi_1(\bar{u}, \bar{a}_z)$ and $\phi_2(\bar{u}, \bar{a}_z)$, respectively. Then it suffices for $\zeta(\bar{a}_z)$ to say that $\phi_1(\bar{x}, \bar{a}_z)$ and $\phi_2(\bar{x}, \bar{a}_z)$ are non-conjugate maximal parabolics, and that the corresponding geometry on the cosets is a generalized polygon. Consider now the non-self-dual case. Let P_i and ϕ_i, for $i = 1, 2$, as before. Then the two conjugacy classes P_1^{Σ} and P_2^{Σ} are definable, and invariant under $\text{Aut}(\Sigma)$ (even for saturated elementary extensions of Σ); for if there was $g \in \text{Aut}(\Sigma)$ interchanging P_1^{Σ} and P_2^{Σ}, this would give an isomorphism from the corresponding polygon to its dual. Thus, for example, by a compactness argument, P_1^{Σ} and P_2^{Σ} are 0-definable, i.e. there are formulas $\psi_i(\bar{x}, \bar{z})$, for $i = 1, 2$, such that:

$$H \in P_1^{\Sigma} \iff H = \psi_1(\Sigma, \bar{b}_1) \quad \text{for some } \bar{b}_1 \in \Sigma^{l(\bar{z})},$$
$$H \in P_2^{\Sigma} \iff H = \psi_2(\Sigma, \bar{b}_2) \quad \text{for some } \bar{b}_2 \in \Sigma^{l(\bar{z})}.$$

Then $\zeta(\bar{a}_z)$ should express that $\phi_1(\Sigma, \bar{a}_z) = \psi_1(\Sigma, \bar{b}_1)$ for some $\bar{b}_1$, $\phi_2(\Sigma, \bar{a}_z) = \psi_2(\Sigma, \bar{b}_2)$ for some $\bar{b}_2$, and that the coset geometry of $\phi_1(\Sigma, \bar{a}_z)$ and $\phi_2(\Sigma, \bar{a}_z)$ is a generalized polygon.

For Lemma 5.14(ii), let σ be the sentence, as in Theorem 7.1, picking out (among finite groups) the members of $\mathcal{G}$; by Remark 5.19, these may be assumed simple. Then, $\eta(\bar{y})$ just says that the little projective group may be interpreted as in Proposition 6.4, and that it is simple and satisfies σ. $\square$

Corollary 7.3. *Let $\mathcal{C}$ be any family of finite Moufang polygons as in Theorem* 6.1, *and let $\mathcal{F}$ be the corresponding class of finite* (*difference*) *fields associated to $\mathcal{C}$. Then, $\mathcal{C}$ is UPI bi-interpretable with $\mathcal{F}$.*

Proof. This follows directly from Theorems 5.18(i) and 7.2(i). □

Proof of Theorem 7.1. The proof is based on [17], where it is shown that each family $\mathcal{G}$ of finite simple groups is UPI bi-interpretable (in fact UPD bi-interpretable) with a family of finite (difference) fields $\mathcal{F}$; we already quoted this as Theorem 5.18(i).

Let $G = G(K)$ be a finite group from the class $\mathcal{G}$, where K denotes the underlying finite (difference) field of G (i.e. for $\mathrm{PSL}_n(q)$ it is $\mathbb{F}_q$, for $\mathrm{PSU}_n(q)$ – a subgroup of $\mathrm{PSL}_n(q^2)$ – it is $\mathbb{F}_q$, for ${}^2F_4$ it is $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$, and so on). We want the sentence σ to describe the following:

(a) a uniform definition of a copy $K^\star$ of K with $K^\star \subseteq G$;
(b) $K^\star \in \mathcal{F}$;
(c) a uniform construction of a copy $G^{\star\star}$ of G, living on a power of G, whose underlying field is exactly $K^\star$;
(d) a uniform definition of an isomorphism $G \longrightarrow G^{\star\star}$.

Since all the cases above are extensively treated in [17], we do not give any detail. Each part of (a)–(d) is dealt with in [17] in two different contexts, namely the untwisted case and the twisted case; also, in the twisted case there are two sub-cases: groups with roots of the same length and groups with roots of different lengths, i.e. Suzuki and Ree groups (see, in particular, Discussion 5.4.1 of [17]). For the Suzuki and Ree groups (i.e. ${}^2B_2$, ${}^2G_2$ and ${}^2F_4$) difference fields, rather than pure fields, are required (i.e. $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$ and $(\mathbb{F}_{3^{2k+1}}, x \mapsto x^{3^k})$).

We first proceed by excluding the case of Suzuki and Ree groups, since for the other families, all but finitely many finite fields arise. Part (a), for the uniform interpretation of $K \in \mathcal{F}$ in G, comes all from Sections 5.2.4, 5.3.4 and 5.4 of [17]. Let $\theta(\bar{x}, \bar{y})$ be a formula, and let $\bar{a}_y$ be a tuple of G such that $\theta(\bar{x}, \bar{a}_y)$ interprets K, as well as its field structure (field addition and multiplication), and denote by $K^\star$ such interpretation of K in G. Part (b) follows from Remark 5.2.9 of [17]; more precisely, it shows that the formula $\theta(\bar{x}, \bar{y})$ can be augmented to a formula $\theta^\star = \theta^\star(\bar{x}, \bar{y})$ interpreting exactly the members of $\mathcal{F}$, i.e. ruling out members of $\mathcal{D} \setminus \mathcal{F}$ by listing their isomorphism types. We can now collect the following: there exists a formula $\eta(\bar{y})$ such that if $\bar{a}_y \in G$, for some $G \in \mathcal{G}$, then $\eta(\bar{a}_y)$ holds if and only if $\theta^\star(\bar{x}, \bar{a}_y)$ interprets a member of $\mathcal{F}$, with $\eta(\bar{y})$ being as in Lemma 5.14(ii). Part (c) is given by Lemmas 5.2.5 and 5.3.5, and Corollary 5.4.3(i) of [17]. Finally, for part (d), Lemma 4.3.10 of [17] tells us how to extend the uniform parameter L_{group}-definable isomorphism between $K^\star$ and $K^{\star\star\star}$, i.e. the re-interpretation of $K^\star$ in itself, to the whole of G, so that we have a uniformly parameter L_{group}-definable isomorphism between G and $G^{\star\star}$.

Let now τ be a sentence which axiomatizes the appropriate class $\mathcal{F}$ of finite fields. Also, let $\phi^\star(\bar{u}, \bar{z})$ interpret $G^{\star\star}$ in $K^\star$, as in part (c). Finally, let $\psi(\bar{x}, \bar{u}, \bar{v})$ be a formula defining an isomorphism from G to $G^{\star\star}$, as in part (d). Then, σ is a first order sentence expressing:

$$\exists \bar{y}\, \exists \bar{z}\, \exists \bar{v} \big(\theta^\star(G, \bar{y}) \models \tau \wedge \phi^\star(\bar{u}, \bar{z}) \wedge \psi(\bar{x}, \bar{u}, \bar{v})\big).$$

This is first order expressible; for example, $\theta^\star \models \tau$ is expressed by relativising the quantifiers in τ to $\{\bar{x} \in G\colon\ \theta^\star(\bar{x}, \bar{y}) \text{ holds}\}$.

A small modification of this argument handles the Suzuki and Ree groups. For example, the class of finite difference fields $(\mathbb{F}_{2^{2k+1}}, x \mapsto x^{2^k})$ can be characterized among all finite difference fields (F, σ), by expressing that $\mathrm{char}(F) = 2$ and $\sigma^2 \circ \mathrm{Frob} = \mathrm{id}$. □

8. Supersimple Moufang polygons

In this section, we extend the methods used above to prove Theorem 8.2 (which yields Theorem 1.2). In the following, by $\Gamma(K)$ we mean a good polygon (see Definition 3.6) coordinatized over K, in the informal meaning of Remark 2.8; likewise, we denote by $\Sigma(K)$ the little projective group associated to $\Gamma(K)$. Notice that, despite Sections 6 and 7, in Theorem 8.2 below $\Sigma(K)$ is not necessarily assumed to be finite; thus, the group structures associated to good Moufang polygons are not necessarily those listed in Table 6.1. However, $\Sigma(K)$ is, essentially (up to the kernel of the action of $\Sigma(K)$ on $\Gamma(K)$), an extension of the group of K-rational points of a simple algebraic group of relative rank 2, a classical group of rank 2, or a group of mixed type; see, for instance, Chapter 41 of [19].

In our final theorem there is a special case concerning the Moufang octagon. We will treat this special case in the next proposition (which appears as Corollary 3.5.2 in [7]), but we will only sketch its proof. First, we need to add a few comments about Moufang octagons. The latter are classified in Section 31 of [19]. The classification depends on some mixed quadrangle Γ' arising from the octagon, and it also depends on a polarity (i.e. automorphism of order two) associated to Γ'. This polarity gives rise to the Tits endomorphism σ (see Definition 5.16) associated with the underlying field K of Γ'. Thus, because of the existence of Γ', K and σ, we usually denote Γ by $\Gamma(K,\sigma)$.

Let $\Gamma = O(K,\sigma)$ be a Moufang octagon. We fix an ordinary suboctagon $A = (x_0, x_1, \dots, x_{15}) \subseteq \Gamma$ and a root $\alpha = (x_0, x_1, \dots, x_8) \subseteq A$. Also, associated to A, we define the root groups U_i corresponding to the roots $\alpha_i = (x_i, x_{i+1}, \dots, x_{i+8})$, for $i = 1, 2, \dots, 8$. Then, by 16.9 and 17.7 of [19], the root groups U_1, U_3, U_5 and U_7 are isomorphic to the additive group $(K,+)$ of K, while the root groups U_2, U_4, U_6 and U_8 are isomorphic to the group $K_\sigma^{(2)} := (K \times K, \cdot)$, where $(t,u)\cdot(s,v) = (t+s+u^\sigma v, u+v)$ for all $(t,u),(s,v) \in K_\sigma^{(2)}$ (the notation is from [19]). For the former, we label the isomorphisms by u_i, for $i = 1,3,5,7$, and denote the elements of U_i by $u_i = u_i(t)$, for each i.

As we did in Section 4, using parameters from A, all the root groups U_i, together with their action on the whole of Γ, are A-definable in Γ. In particular, we can define (inside the little projective group associated to Γ) the group $U_{[1,8]} := \langle U_1, U_2, \dots, U_8\rangle$, which by 5.5 and 5.6 of [19] is in bijection with the product $U_1 \times U_2 \times \cdots \times U_8$; also, by restricting to the units $1_{U_i} \in U_i$ for $i = \{2,4,6,8\}$, we can define $U'_{[1,4]} := \langle U_1, U_3, U_5, U_7\rangle \leqslant U_{[1,8]}$.

In Sections 7 and 8 of [19] it is shown how to construct from the definable quintuple $(U'_{[1,4]}, U_1, U_3, U_5, U_7)$ a Moufang quadrangle $\Gamma' = \Gamma'(K)$, say, which is of mixed type by 31.8 of [19] and whose underlying field is still K. Since this construction is first order definable, Γ' is then definable in Γ.

Proposition 8.1. *Let $\Gamma = O(K,\sigma)$ be a Moufang octagon. Then σ is definable in Γ.*

Sketch of the proof. By 6.1 of [19], there exist unique functions μ_i which fix, for $i = 1,2,\dots,8$, x_i and x_{i+8}, reflect A, and satisfy $U_j^{\mu_i(u_i)} = U_{2i+8-j}$ for each $u_i \in U_i \backslash \{1_{U_1}\}$ and each j. By the uniqueness, μ_i are A-definable elements of $U_{[1,8]} := \langle U_1, U_2, \dots, U_8\rangle$. By 31.9(i) of [19] there exists an element $e_8 \in U_8 \backslash \{1_{U_8}\}$ such that $\mu_8(e_8)^2 = 1$. The element e_8 will play the role of a parameter. It follows from 6.1 of [19] that $U_i^{\mu_8(e_8)} = U_{8-i}$ for every $i \in \{1,3,5,7\}$; therefore $\mu_8(e_8)$ acts on the mixed quadrangle Γ' associated with Γ. Put $\alpha := \mu_8(e_8)$. Then the action of α can be extended to an automorphism of order two of Γ'. It then follows from 24.6 of [19] that there exists an endomorphism ϕ of K such that (K,ϕ) is an octagonal set (i.e. a field K of characteristic 2 equipped with a Tits endomorphism), that $u_3(t)^\alpha = u_2(t^\phi)$ for all $t \in K$, and that ϕ can be extended to the whole of Γ so that $\phi = \sigma$. Since α is definable, from the equation $u_3(t)^\alpha = u_2(t^\sigma)$ it follows that σ is definable in Γ. $\square$

Theorem 8.2. *Let $\Gamma(K)$ be a good Moufang generalized n-gon, and let also $\Sigma(K)$ be its associated little projective group. Then*:

(i) *$\Gamma(K)$ and $\Sigma(K)$ are bi-interpretable* (*with parameters*).

In particular:

(ii) *$\Gamma(K)$ is supersimple finite rank if and only if $\Sigma(K)$ is supersimple finite rank.*
(iii) *If $\Gamma(K)$ is measurable, then K is weakly measurable.*
(iv) *If $\Sigma(K)$ is measurable, then $\Gamma(K)$ is weakly measurable.*
(v) *If $\Sigma(K)$ is pseudofinite, then $\Gamma(K)$ is measurable.*

Remark 8.3. 'Weakly measurable' can probably be strengthened in (iii) and (iv) to measurable, using an analogue of Definition 5.11. The work has not yet been done.

Proof of Theorem 8.2. First, notice that (ii) and (iv) follow from (i). For (iii), we can appeal to Lemmas 3.1, 3.3 and 4.9 of [13], where it is shown how to define the field K from a Moufang polygon $\Gamma(K)$, provided that some conditions on the associated little projective group $\Sigma(K)$ are satisfied; since all good Moufang polygons satisfy the assumptions required by these lemmas, part (iii) follows. Moreover, in the particular case of a Ree–Tits octagon $O(K,\sigma)$, Proposition 8.1 shows that also σ is definable in $\Gamma(K)$; hence, in (iii), if $\Gamma(K)$ is a measurable Ree–Tits octagon, then (K,σ) is weakly measurable.

For (v), if $\Sigma(K)$ is pseudofinite, then by the main theorem of [22] it is elementarily equivalent to a non-principal ultraproduct of a class $\mathcal{G}$ of either finite Chevalley groups of a fixed type or finite twisted groups of fixed Lie type and Lie rank. Thus, by Los' theorem, the associated good Moufang polygon $\Gamma(K)$ interpreted in $\Sigma(K)$ is also elementarily equivalent to a non-principal ultraproduct of a class $\mathcal{C}$ of finite structures; namely, $\mathcal{C}$ is a class of finite Moufang polygons. Therefore, $\mathcal{C}$ is an asymptotic class and, by Theorem 7.2 and Proposition 5.6, $\Gamma(K)$ is measurable.

To prove (i), let $\Gamma = \Gamma(K)$ be a good Moufang polygon and let $\Sigma = \Sigma(K)$ be its corresponding little projective group. For the interpretation of Σ in Γ, it is done exactly as in the proof of Proposition 6.4, by appealing to results from [3]. To interpret Γ in Σ, we also follow [3]. Here we have to distinguish between the self-dual and non-self-dual cases, but this is addressed exactly as in the proof of Theorem 7.2(i); thus, we omit it and refer back to Theorem 7.2 for the details about the non-self-dual case. First, in Γ, let $A = (x_0, x_1, \ldots, x_{2n-1})$ be a fixed ordinary polygon, $\alpha = (x_0, x_1, \ldots, x_n)$ a fixed root in A, and $x_0 I x_{2n-1}$ a fixed flag in A. Also, let B be the stabilizer (in Σ) of $x_0 I x_{2n-1}$ and N be the setwise stabilizer (in Σ) of A; then, as in 33.4 of [19], Σ has a BN-pair. With the notation of [3], let now $\mathcal{P}^{(B)} := \{P_J = U_J L_J\colon\ J \subseteq I\}$ be the set of maximal parabolic subgroups of Σ containing B. It then follows from the argument used in the proof of Theorem 5.3 of [13] that the elements of $\mathcal{P}^{(B)}$ are uniformly definable (notice that the mentioned argument makes use of a finite rank assumption, but we do not need it since in our case the group is already the little projective group, and therefore we can use its minimality properties without directly referring to Zilber Indecomposibility arguments). Hence, since every parabolic subgroup is an intersection of finitely many maximal parabolics, it follows that we can interpret Γ from $\mathcal{P}^{(B)}$; see Section 15.5 of [3] (it deals with buildings, but by Theorem 2.10(i) the Tits rank 2 case gives exactly the construction of generalized polygons). Finally, for the definability of the isomorphisms $g^{\star} f$ and $f^{\star} g$ (with the notation of Definition 5.8) we can essentially proceed as done in Lemmas 6.6 and 6.7 for the finite case; we omit the details. $\square$

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